36. SOLVING ABSOLUTE VALUE INEQUALITIES INVOLVING "LESS THAN"

solving inequalities involving absolute value

This section should feel remarkably similar to the previous one. Instead of solving absolute value *equations*, this section presents the tools needed to solve absolute value *inequalities* involving 'less than,' like these:

$$|x| < 5$$
$$|x| \le 3$$
$$|2 - 3x| < 7$$

Each of these inequalities has only a single set of absolute value symbols which is by itself on the left-hand side of the sentence, and has a *variable* inside the absolute value. The verb is either '$<$' or '$\le$'. As in the previous section, solving sentences like these is easy, if you remember the critical fact that

absolute value gives distance from 0.

Keep this in mind as you read the following theorem:

THEOREM

tool for solving absolute value inequalities involving 'less than'

Let $x \in \mathbb{R}$, and let $k > 0$. Then,

$$|x| < k \iff -k < x < k$$
$$|x| \le k \iff -k \le x \le k$$

$|x| < k$ is an entire class of sentences

Recall first that normal mathematical conventions dictate that '$|x| < k$' and '$|x| \le k$' represent entire classes of sentences, including $|x| < 2$, $|x| \le 5.7$, and $|x| < \frac{1}{3}$. The variable k changes from sentence to sentence, but is constant within a given sentence.

EXERCISES

1. a. Give three sentences of the form '$|x| < k$' where $k > 0$. Use examples different from those given above.
   b. Give three sentences of the form '$-k < x < k$' where $k > 0$.

translating the theorem:

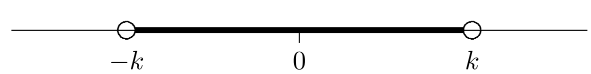
thought process for solving sentences like $|x| < k$

When you see a sentence of the form '$|x| < k$', here's what you should do:

- Check that k is a positive number.
- The symbol $|x|$ represents the distance between x and 0.
- Thus, you want the numbers x, whose distance from 0 is less than k.

$$\overbrace{|x|}^{\text{want numbers } x \text{, whose distance from } 0} \qquad \overbrace{<}^{\text{is less than}} \qquad \overbrace{k}^{k}$$

- You can walk from 0 in two directions: less than k units to the left, or less than k units to the right. So, you want all the numbers between $-k$ and k.



$|x| < k$

- Thus, $|x| < k$ is equivalent to $-k < x < k$.

filling in some blanks to help your thought process

When you see a sentence like '$|x| < 7$', you thought process should be like filling in the following blanks:

We want the numbers _____ , whose _____ from _____ is less than _____ . Thus, we want _____ to be between _____ and _____ .

The correctly-filled-in blanks are:

We want the numbers <u>x</u> , whose <u>distance</u> from <u>0</u> is less than <u>7</u> . Thus, we want <u>x</u> to be between <u>-7</u> and <u>7</u> .

EXERCISES

2. Fill in the blanks:
   a. When you look at the sentence '$|x| < 5$', you should think: We want the numbers _____ , whose _____ from _____ is less than _____ . Thus, we want _____ to be between _____ and _____ .
   b. When you look at the sentence '$|z| < \frac{1}{5}$', you should think: We want the numbers _____ , whose _____ from _____ is less than _____ . Thus, we want _____ to be between _____ and _____ .
   c. When you look at the sentence '$|x| < k$' (with $k > 0$), you should think: We want the numbers _____ , whose _____ from _____ is less than _____ . Thus, we want _____ to be between _____ and _____ .
3. Give a sentence, not using absolute value symbols, that is equivalent to:
   a. $|x| < 3$
   b. $|t| \le 4.2$
4. Give a sentence, using absolute value symbols, that is equivalent to:
   a. $-7 < x < 7$
   b. $-\frac{1}{3} \le t \le \frac{1}{3}$
5. Give a precise mathematical statement of the tool that says that a sentence like '$|x| < 5$' can be transformed to the equivalent sentence '$-5 < x < 5$'.
6. Is the sentence '$|x| < -6$' of the form described in the previous theorem? Why or why not?
7. Can the sentence '$|x| - 5 < 7$' be transformed to a sentence of the form described in the previous theorem? If so, what is the equivalent sentence?

EXAMPLE

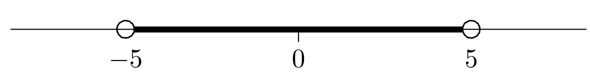
solving a sentence of the form $|x| < k$

Example: Solve: $|x| < 5$

Solution:

$$|x| < 5$$
$$-5 < x < 5$$

The solution set is shaded below. Be sure to put hollow dots at 5 and -5 to indicate that these endpoints are not included.



EXAMPLE

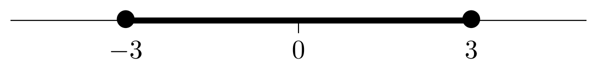
solving a sentence of the form $|x| \le k$

Example: Solve: $|x| \le 3$

Solution:

$$\begin{aligned} |x| &\le 3 \\ -3 \le x &\le 3 \end{aligned}$$

The solution set is shaded below. Be sure to put solid dots at 3 and -3 to indicate that these endpoints are included.



solving more complicated sentences of the form $|x| < k$ *;*

x can be ANYTHING!

The power of the tool

$$|x| < k \iff -k < x < k$$

goes way beyond solving simple sentences like '$|x| < 5$'. Since x can be *any* real number, you should think of x as merely representing *the stuff inside the absolute value symbols*. Thus, you could think of rewriting the tool as:

$$|\text{stuff}| < k \iff -k < \text{stuff} < k$$

Thus, we have all the following equivalences:

$$\begin{aligned} |\overbrace{2-3x}^{\text{stuff}}| < 7 &\iff -7 < \overbrace{2-3x}^{\text{stuff}} < 7 \\ |\overbrace{5x-1}^{\text{stuff}}| < 8 &\iff -8 < \overbrace{5x-1}^{\text{stuff}} < 8 \\ |\overbrace{x^2-3x+4}^{\text{stuff}}| < \frac{1}{5} &\iff -\frac{1}{5} < \overbrace{x^2-3x+4}^{\text{stuff}} < \frac{1}{5} \end{aligned}$$

and so on!

Of course, this idea works with the verb '$\le$' as well as '$<$'.

EXERCISES

8. For each of the following, write an equivalent sentence that does not use absolute value symbols. Do *not* solve the resulting sentences.
   a. $|1 + 2x| < 3$
   b. $|7x - \frac{1}{2}| \le 5$
   c. $|x^2 - 8| < 0.4$

getting ready to solve more complicated inequalities; using transformations on sentences of the form $a < x < b$

Before solving more complicated absolute value inequalities, we need to pause and talk about the solution of sentences like '$-9 < -3x < 5$'.

Recall that the compound inequality '$-9 < -3x < 5$' is equivalent to the 'and' sentence

$$-9 < -3x \quad \text{and} \quad -3x < 5\,.$$

the hard way—break into an 'and' sentence

Recall also that when both sides of an inequality are multiplied or divided by a negative number, then the direction of the verb must change. Thus, we *could* transform the sentence '$-9 < -3x < 5$' by breaking it into an 'and' sentence, like this:

$-9 < -3x < 5$	original sentence
$-9 < -3x \quad \text{and} \quad -3x < 5$	break into an 'and' sentence
$3 > x \quad \text{and} \quad x > -\frac{5}{3}$	divide by -3; verbs change
$-\frac{5}{3} < x \quad \text{and} \quad x < 3$	rewrite sentences right-to-left, re-order
$-\frac{5}{3} < x < 3$	combine into compound inequality

the easy way—don't break into an 'and' sentence

In practice, however, it's tedious and unnecessary to break the compound inequality into an 'and' sentence. Instead, just apply the required transformations across the entire sentence, like this:

$-9 < -3x < 5$	original sentence
$3 > x > -\frac{5}{3}$	divide by -3, change both verbs
$-\frac{5}{3} < x < 3$	rewrite in conventional way

The process is much shorter this way!

EXERCISES

9. Transform each sentence to the form '$a < x < b$' or '$a \le x \le b$', where $a < b$. Do NOT break into an 'and' sentence.
   a. $8 < 2x < 14$
   b. $-9 < 3x < 12$
   c. $-15 < -5x < 10$
   d. $0 \le -2x \le 4$
   e. $4 \le -2x + 3 \le 7$

EXAMPLE

Example: Solve: $|2-3x| < 7$

Solution: Be sure to write a nice, clean list of equivalent sentences.

$|2-3x| < 7$	original sentence
$-7 < 2-3x < 7$	$|x| < k \iff -k < x < k$
$-9 < -3x < 5$	subtract 2
$3 > x > -\frac{5}{3}$	divide by -3; change verbs
$-\frac{5}{3} < x < 3$	rewrite in conventional way

doing a 'spot-check'

There are infinitely many real numbers between $-\frac{5}{3}$ and 3, so it's impossible to check all the solutions to '$-\frac{5}{3} < x < 3$'. Instead, it's common to do a 'spot-check,' as follows.

To 'spot-check,' choose a value of x for which the final sentence is true, as well as a value of x for which the final sentence is false. Choose easy values to work with. Substitute them into the original sentence, and check that it is also true and false. These 'spot-checks' catch a lot of silly mistakes.

a value of x for which the final sentence is true

For example, the sentence '$-\frac{5}{3} < x < 3$' is true when $x = 0$. Substituting $x = 0$ into the original sentence '$|2-3x| < 7$' gives:

$$|2-3(0)| \overset{?}{<} 7$$
$$2 < 7 \text{ is true}$$

a value of x for which the final sentence is false

The sentence '$-\frac{5}{3} < x < 3$' is false when $x = 4$. Substituting $x = 4$ into the original sentence '$|2-3x| < 7$' gives:

$$|2-3(4)| \overset{?}{<} 7$$
$$10 < 7 \text{ is false}$$

Some people prefer to check *two* false values, one on each side. The sentence '$-\frac{5}{3} < x < 3$' is also false when $x = -2$. Substituting $x = -2$ into the original sentence '$|2-3x| < 7$' gives:

$$|2-3(-2)| \overset{?}{<} 7$$
$$8 < 7 \text{ is false}$$

Of course, you can also check by going over your work again, step-by-step.

an alternate checking approach:

check the endpoints of the solution interval

It has been shown that the inequality '$|2-3x| < 7$' is equivalent to '$-\frac{5}{3} < x < 3$'; thus, the solution set to '$|2-3x| < 7$' is the interval $(-\frac{5}{3}, 3)$. When the endpoints of this interval are substituted into the original sentence, the same number should appear on both sides. This gives an alternate checking approach:

$\lvert 2-3(-\frac{5}{3})\rvert < 7$	substitute left-hand endpoint into original sentence
$\lvert 2+5\rvert < 7$	simplify
$7 < 7$	same number results on both sides
$\lvert 2-3(3)\rvert < 7$	substitute right-hand endpoint into original sentence
$\lvert 2-9\rvert < 7$	simplify
$7 < 7$	same number results on both sides

Notice that, in both cases, the resulting sentence '$7 < 7$' is false, because the numbers $-\frac{5}{3}$ and 3 are not in the solution set.

EXERCISES

10. Solve. Write a nice, clean list of equivalent sentences.
    a. $|1+2x| < 3$ (Check by doing several 'spot-checks'.)
    b. $|7x-5| \le 2$ (Check the endpoints of the solution interval.)
    c. $|4-9x| < 1$ (Use your favorite checking method.)

What happens if k is negative in the sentence '$|x| < k$'?

What about a sentence like '$|x| < -5$', where the absolute value is less than a negative number? Notice that this situation is not covered in the theorem, since the sentence '$|x| < k$' is only addressed with $k > 0$.

Recall that $|x| \ge 0$ for all real numbers x. Thus, the sentence '$|x| < -5$' is never true. The expression $|x|$ represents a *distance from zero*, and distances cannot be negative.

first step when analyzing $|x| < k$: check that $k > 0$

Whenever you're working with a sentence of the form '$|x| < k$', you must always check first that $k > 0$. If k is negative, you just stop and say that the sentence is always false. Here are some examples, which illustrate different ways that you can state your answer:

'$|x| < -3$' is always false.

'$|2x-1| \le -5$' is never true.

'$|3x-5x^2+7| < -0.4$' has an empty solution set.

applying transformations to isolate the absolute value on the left-hand side

Observe that in the model '$|x| < k$', the absolute value is all by itself on the left-hand side. In mathematics, something all by itself on one side of a sentence is said to be *isolated*. Thus, getting the absolute value all by itself on the left-hand side is usually phrased like this: *isolate the absolute value on the left-hand side.*

Whenever you solve any absolute value sentence, you should begin by isolating the absolute value on the left-hand side, as the next example illustrates.

EXAMPLE

Example: Solve: $6 < -2|1-5x|$

Solution:

$6 < -2\|1-5x\|$	original sentence
$-3 > \|1-5x\|$	divide by -2; change verbs
$\|1-5x\| < -3$	rewrite; stop—sentence is always false

What if k is zero?

If k is zero, stop and investigate the particular situation. An absolute value can't be *less* than zero, but it can *equal* zero. Here are some examples:

Example: $|x| < 0$ is always false, since $|x|$ cannot be less than 0.

Example: $|x| \le 0$ is true when $x = 0$.

Example: $|3x+5| < 0$ is always false, since absolute value cannot be less than 0.

Example: $|3x+5| \le 0$ is true when $3x+5=0$, thus when $x = -\frac{5}{3}$.

EXERCISES

11. Transform into an equivalent sentence with the absolute value isolated on the left-hand side. Solve the resulting sentence. If the sentence is always false, so state. Report the solution set using correct set notation.
    a. $3 > |3x-7| + 4$
    b. $-4 > |2x-5|$
    c. $2 > |5+8x|$
    d. $|7x-2| - 4 \le -3$
    e. $|9x+2| \le 0$
    f. $0 > |6x+5|$

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: SOLVING ABSOLUTE VALUE INEQUALITIES INVOLVING "LESS THAN"

1. a. $|x| < 1$, $|x| < 7.2$, and $|x| < \frac{1}{2}$

b. $-1 < x < 1$, $-7.2 < x < 7.2$, and $-\frac{1}{2} < x < \frac{1}{2}$

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2. a. We want the numbers $\underline{x}$, whose <u>distance</u> from $\underline{0}$ is less than $\underline{5}$. Thus, we want $\underline{x}$ to be between $\underline{-5}$ and $\underline{5}$.

b. We want the numbers $\underline{z}$, whose <u>distance</u> from $\underline{0}$ is less than $\underline{\frac{1}{5}}$. Thus,we want $\underline{z}$ to be between $\underline{-\frac{1}{5}}$ and $\underline{\frac{1}{5}}$.

c. We want the numbers $\underline{x}$, whose <u>distance</u> from $\underline{0}$ is less than $\underline{k}$. Thus, we want $\underline{x}$ to be between $\underline{-k}$ and $\underline{k}$.

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3. a. $|x| < 3$ is equivalent to $-3 < x < 3$

b. $|t| \le 4.2$ is equivalent to $-4.2 \le t \le 4.2$

4. a. $-7 < x < 7$ is equivalent to $|x| < 7$

'$-\frac{1}{3} \le t \le \frac{1}{3}$' is equivalent to $|t| \le \frac{1}{3}$

5. For all real numbers x , and for $k \ge 0$, $|x| < k$ is equivalent to $-k < x < k$.

6. The sentence '$|x| < -6$' is not of the form in the theorem, because -6 is a negative number.

7. The sentence '$|x| - 5 < 7$' can be transformed to '$|x| < 12$' by adding 5 to both sides.

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8. a. $|1 + 2x| < 3$ is equivalent to $-3 < 1 + 2x < 3$

b. $|7x - \frac{1}{2}| \le 5$ is equivalent to $-5 \le 7x - \frac{1}{2} \le 5$

c. $|x^2 - 8| < 0.4$ is equivalent to $-0.4 < x^2 - 8 < 0.4$

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9. a.

$$\begin{array}{ll} 8 < 2x < 14 & \text{original sentence} \\ 4 < x < 7 & \text{divide by } 2 \end{array}$$

b.

$$\begin{array}{ll} -9 < 3x < 12 & \text{original sentence} \\ -3 < x < 4 & \text{divide by } 3 \end{array}$$

c.

$$\begin{array}{ll} -15 < -5x < 10 & \text{original sentence} \\ 3 > x > -2 & \text{divide by } -5\text{, change verbs} \\ -2 < x < 3 & \text{rewrite in conventional way} \end{array}$$

d.

$$\begin{array}{ll} 0 \le -2x \le 4 & \text{original sentence} \\ 0 \ge x \ge -2 & \text{divide by } -2\text{, change verbs} \\ -2 \le x \le 0 & \text{rewrite in conventional way} \end{array}$$

e.

$4 \le -2x + 3 \le 7$	original sentence
$1 \le -2x \le 4$	subtract 3
$-\frac{1}{2} \ge x \ge -2$	divide by -2, change verbs
$-2 \le x \le -\frac{1}{2}$	rewrite in conventional way

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10. a.

$\lvert 1 + 2x\rvert < 3$	original sentence
$-3 < 1 + 2x < 3$	$\lvert x\rvert < k \iff -k < x < k$
$-4 < 2x < 2$	subtract 1
$-2 < x < 1$	divide by 2

Spot-checks:
Choose a simple number between -2 and 1; the number $x = 0$ makes the final sentence true:

$$\lvert 1 + 2(0)\rvert \overset{?}{<} 3$$
$$1 < 3 \ \text{ is true}$$

Choose a simple number greater than 1; the number $x = 2$ makes the final sentence false:

$$\lvert 1 + 2(2)\rvert \overset{?}{<} 3$$
$$5 < 3 \ \text{ is false}$$

Choose a simple number less than -2; the number $x = -3$ makes the final sentence false:

$$\lvert 1 + 2(-3)\rvert \overset{?}{<} 3$$
$$\lvert 1 - 6\rvert \overset{?}{<} 3$$
$$5 < 3 \ \text{ is false}$$

b.

$\lvert 7x - 5\rvert \le 2$	original sentence
$-2 \le 7x - 5 \le 2$	$\lvert x\rvert \le k \iff -k \le x \le k$
$3 \le 7x \le 7$	add 5
$\frac{3}{7} \le x \le 1$	divide by 7

Checking the endpoints of the solution interval:

$$\lvert 7(\tfrac{3}{7}) - 5\rvert \le 2$$
$$\lvert 3 - 5\rvert \le 2$$
$$2 \le 2$$

$$\lvert 7(1) - 5\rvert \le 2$$
$$2 \le 2$$

In both cases, the same number is obtained on both sides. Also, the sentence '$2 \le 2$' is true, since the numbers $\frac{3}{7}$ and 1 are in the solution set.

c.

$\|4 - 9x\| < 1$	original sentence
$-1 < 4 - 9x < 1$	$\|x\| < k \iff -k < x < k$
$-5 < -9x < -3$	subtract 4
$\frac{5}{9} > x > \frac{1}{3}$	divide by -9; change verbs
$\frac{1}{3} < x < \frac{5}{9}$	rewrite in conventional way

Checking the endpoints of the solution interval:

$$\begin{gathered} |4 - 9(\frac{1}{3})| < 1 \\ |4 - 3| < 1 \\ 1 < 1 \end{gathered}$$

$$\begin{gathered} |4 - 9(\frac{5}{9})| < 1 \\ |4 - 5| < 1 \\ |-1| < 1 \\ 1 < 1 \end{gathered}$$

In both cases, the same number is obtained on both sides. Also, the sentence '$1 < 1$' is false, since the numbers $\frac{1}{3}$ and $\frac{5}{9}$ are not in the solution set.

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11. a.

$3 > \|3x - 7\| + 4$	original sentence
$-1 > \|3x - 7\|$	subtract 4
$\|3x - 7\| < -1$	rewrite; stop—sentence is always false

The solution set is empty.

b.

$-4 > \|2x - 5\|$	original sentence
$\|2x - 5\| < -4$	rewrite; stop—sentence is always false

The solution set is empty.

c.

$2 > \|5+8x\|$	original sentence
$\|5+8x\| < 2$	rewrite
$-2 < 5+8x < 2$	$\|x\| < k \iff -k < x < k$
$-7 < 8x < -3$	subtract 5
$-\frac{7}{8} < x < -\frac{3}{8}$	divide by 8

The solution set is the interval $(-\frac{7}{8}, -\frac{3}{8})$.

d.

$\|7x-2\| - 4 \le -3$	original sentence
$\|7x-2\| \le 1$	add 4
$-1 \le 7x-2 \le 1$	$\|x\| \le k \iff -k \le x \le k$
$1 \le 7x \le 3$	add 2
$\frac{1}{7} \le x \le \frac{3}{7}$	divide by 7

The solution set is the interval $[\frac{1}{7}, \frac{3}{7}]$.

e.

$\|9x+2\| \le 0$	original sentence
$9x+2 = 0$	$\|x\| \le 0 \iff x = 0$
$9x = -2$	subtract 2
$x = -\frac{2}{9}$	divide by 9

The solution set must be reported using list notation: $\{-\frac{2}{9}\}$.

f.

$0 > \|6x+5\|$	original sentence
$\|6x+5\| < 0$	rewrite; stop—sentence is always false

The solution set is empty.