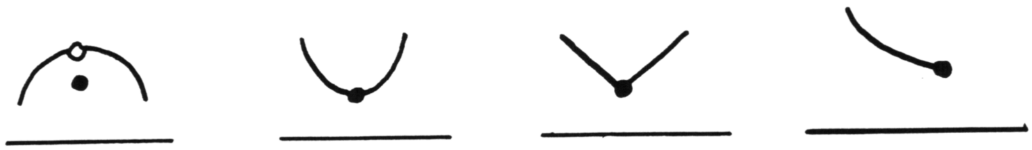
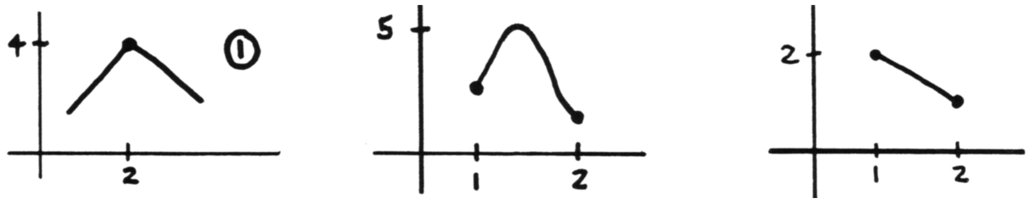
SECTION 5.2 Local Maxima and Minima—Critical Points

IN-SECTION EXERCISES:

EXERCISE 1.



EXERCISE 2.

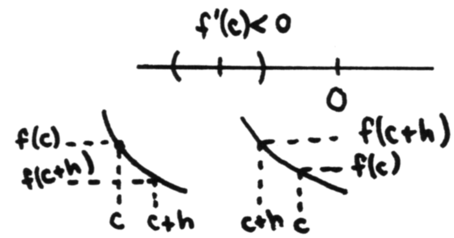


EXERCISE 3.

The function f can NOT have a local maximum at $x = 2$; the stated condition tells us that the sentence '$f(x) \le 3$' is not true on any interval containing $x = 2$. However, there could be a local minimum at $x = 2$.

EXERCISE 4.

Proof. Assume that all the stated conditions are true.



Since $f(c) < 0$:

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} < 0$$

So choose δ so small that whenever $|h - 0| < \delta$, one has:

$$\frac{f(c+h) - f(c)}{h} < 0 \quad (*)$$

By hypothesis, c is *not* an endpoint of the domain of f, so the function is defined both to the right and left of c. Now, whenever $h > 0$ and $|h| < \delta$, multiplying both sides of (*) by the positive number h yields the equivalent inequality

$$f(c+h) - f(c) < 0 \ ,$$

so that $f(c+h) < f(c)$. That is, *arbitrarily close to* $(c, f(c))$, on the right, is another point $(c+h, f(c+h))$ with a *lesser* function value. Thus, $f(c)$ is NOT a local minimum.

Similarly, whenever $h < 0$ and $|h| < \delta$, multiplying both sides of (*) by the negative number h yields the equivalent inequality

$$f(c+h) - f(c) > 0 \ ,$$

so that $f(c+h) > f(c)$. That is, *arbitrarily close to* $(c, f(c))$, *on the left, is another point* $(c+h, f(c+h))$ *with a greater function value*. Thus, $f(c)$ is NOT a local maximum, either.

Thus, f does not have a local extreme value at $x = c$.

EXERCISE 5.

1. Choose P to be false, and Q to be true.
2. Choose P to be true, and Q to be false.
3. Choose P true and Q true; or P false and Q false. In either case, both $P \implies Q$ and $Q \implies P$ are true.

4. If the hypothesis '$x = 2$' is false, then the implication is vacuously true. Thus, it need only be shown that whenever '$x = 2$' is true, so is '$x^2 = 4$'.

   Let $x = 2$. Then, $x^2 = 2^2 = 4$. Thus, the implication is true.

5. The sentence

$$x^2 = 4 \implies x = 2$$

   is an abbreviation for:

$$\text{For all } x,\ \ x^2 = 4 \implies x = 2$$

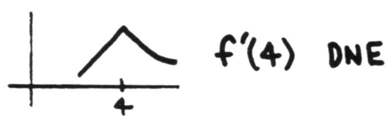
   To show that this 'for all' sentence is false, we need only exhibit one value of x for which '$x^2 = 4 \implies x = 2$' is false.

   Choose $x = -2$. Then, the hypothesis '$(-2)^2 = 4$' is true, but the conclusion '$-2 = 2$' is false. Thus, the implication is false.

EXERCISE 6.

All of the sentences are implicit 'for all' sentences; they should all begin with, 'For all functions f'.

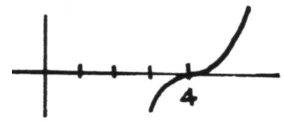
1. TRUE. Every extreme value MUST OCCUR at a critical point.
2. FALSE. The function f graphed below has a local maximum at $x = 4$ (so the hypothesis is true), but $f'(4) \neq 0$ (the conclusion is false).



3. TRUE. The point $(4, f(4))$ must be a critical point. Since f is differentiable everywhere, $f'(4)$ exists. Since $\mathcal{D}(f') = \mathbb{R}$, the domain of f must also be $\mathbb{R}$. Thus, 4 cannot be an endpoint of the domain of f. Thus, it must be that there is a horizontal tangent line at $x = 4$.
4. FALSE. The function f graphed below has a horizontal tangent line at $x = 1$ (the hypothesis is true), but the point $(1, f(1))$ is not a local extreme point for f (the conclusion is false).

EXERCISE 7.

1. Note that the graph of f is the same as the graph of $y = x^3$, except shifted 4 units to the right. The following results confirm this observation.

   Since $\mathcal{D}(f) = [1, \infty)$, the point $(1, f(1)) = (1, -27)$ is a critical point.

   $f'(x) = 3(x-4)^2(1) = 3(x-4)^2$

   $f'(x) = 0 \iff x = 4$; the point $(4, f(4)) = (4, 0)$ is a critical point. These are the only critical points.

   The sketch confirms that $(1, -27)$ is a local minimum; the point $(4, 0)$ is not a local extreme point.

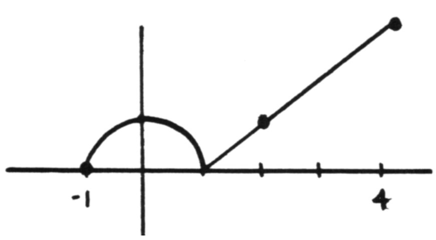
2. First, make a sketch. The sketch shows that f is continuous at $x = 1$, but not differentiable there.

   The points $(-1, 0)$ and $(4, 3)$ are critical points, since they are endpoints of $\mathcal{D}(f)$.

   The point $(1, 0)$ is a critical point, because $f'(0)$ does not exist.

   The point $(0, 1)$ is a critical point, because $f'(0) = 0$.

   The sketch shows that $(-1, 0)$ and $(1, 0)$ are local minima. The points $(0, 1)$ and $(4, 3)$ are local maxima.

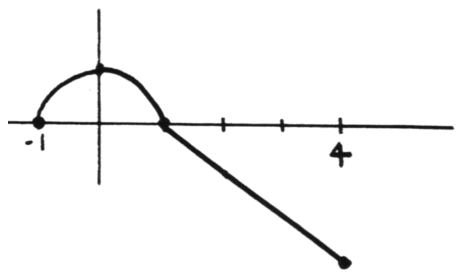


3. A quick sketch shows that f is continuous at $x = 1$. Is f differentiable here? Coming in from the left, the 'directions' are given by $\frac{d}{dx}(-x^2+1) = -2x$; and as x approaches 1, $-2x$ approaches -2 .

   Coming in from the right, the 'directions' are all -1, since $y = -x + 1$ is a line with slope -1. Thus, the directions do NOT 'match up' at $x = 1$.

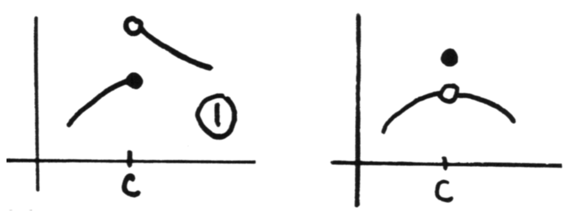
   Thus, $(1, 0)$ is a critical point, since $f'(1)$ does not exist.

   The remaining critical points, and local extreme behavior, are summarized in the chart below.



c	f(c)	WHY?	LOCAL EXT?
-1	0	ENDPT	MIN
0	1	f'(c)=0	MAX
1	0	f'(c) DNE	NO
4	-3	ENDPT	MIN

EXERCISE 8.



EXERCISE 9.

Suppose that c is an endpoint of the domain of f, f is continuous at c, and $f'(x) < 0$ on some interval to the right of c. Then, $(c, f(c))$ is a local maximum for f.

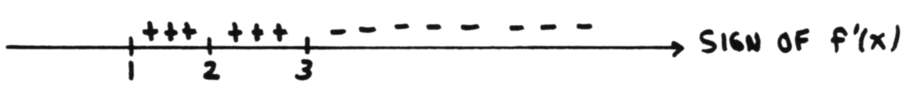
Suppose that c is an endpoint of the domain of f, f is continuous at c, and $f'(x) > 0$ on some interval to the left of c. Then, $(c, f(c))$ is a local maximum for f.

Suppose that c is an endpoint of the domain of f, f is continuous at c, and $f'(x) < 0$ on some interval to the left of c. Then, $(c, f(c))$ is a local minimum for f.

EXERCISE 10.

Recall that $f'(x) = 0 \iff x = 3$, and f' is not defined when $x = 2$. (That is, there is a discontinuity in the graph of f' at $x = 2$.) Thus, there are three intervals to test.

Test Points: $f'(\frac{3}{2}) > 0$; $f'(\frac{5}{2}) > 0$; $f'(4) < 0$



By the First Derivative Test, there is a local minimum at $x = 1$, no local extreme value at $x = 2$, and a local maximum at $x = 3$. This is (of course) in agreement with earlier results.

EXERCISE 11.

1. Note that $\mathcal{D}(f) = [0,1) \cup (1,4)$. All critical points must come from the domain of f.

   The endpoints of $\mathcal{D}(f)$ are critical points: $(0, \frac{e^0}{0-1}) = (0,-1)$ and $(4, \frac{e^4}{4-1}) \approx (4, 18.2)$

   Find f', and determine where $f'(x) = 0$:

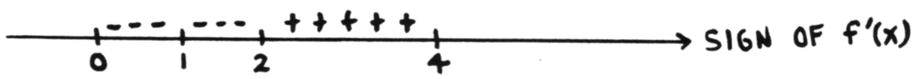
$$\begin{aligned} f'(x) &= \frac{(x-1)e^x - e^x(1)}{(x-1)^2} \\ &= \frac{e^x(x-2)}{(x-1)^2} \\ f'(x) = 0 &\iff x - 2 = 0 \iff x = 2 \end{aligned}$$

   Note that $\mathcal{D}(f') = \mathcal{D}(f)$. The discontinuity at $x = 1$ must be taken into account when determining the sign of $f'(x)$.

   The point $(2, \frac{e^2}{2-1}) \approx (2, 7.4)$ is a critical point; there is a horizontal tangent line here.

   Test Points: $f'(\frac{1}{2}) = \frac{(+)(-)}{(+)} < 0$; $f'(\frac{3}{2}) = \frac{(+)(-)}{(+)} < 0$; $f'(3) = \frac{(+)(+)}{(+)} > 0$

   The sign of $f'(x)$ is summarized below:



   Use the First Derivative Test to make the conclusions summarized below:

c	f(c)	WHY?	LOCAL EXTREMUM?
0	-1	ENDPT	MAX
4	≈ 18.2	ENDPT	MAX
2	≈ 7.4	f'(c) = 0	MIN

2. Note that $\mathcal{D}(f) = [0, 8]$. All critical points must come from the domain of f.

   The endpoints of $\mathcal{D}(f)$ are critical points: $(0, 0)$ and $(8, 514)$

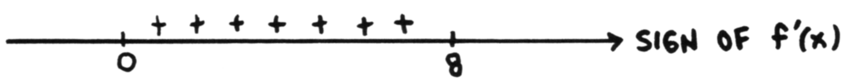
   Find f', and determine where $f'(x) = 0$:

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-2/3} + 3x^2 \\ &= \frac{1}{3\sqrt[3]{x^2}} + 3x^2 \\ &= \frac{1 + 9\sqrt[3]{x^8}}{3\sqrt[3]{x^2}} \\ f'(x) = 0 &\iff 9\sqrt[3]{x^8} = -1 \end{aligned}$$

   Since $x^8 \geq 0$, we see that $f'(x)$ never equals zero.

   Note that f' is not defined when $x = 0$, even though f IS defined at 0. Thus, $(0, 0)$ is (for a second reason!) a critical point.

   Test Point: $f'(1) > 0$



   Use the First Derivative Test to make the conclusions summarized below:

c	f(c)	WHY?	LOCAL EXT?
0	0	ENDPT	MIN
8	514	ENDPT	MAX

END-OF-SECTION EXERCISES:

1. In Exercise 5, question 4, the sentence '$x = 2 \implies x^2 = 4$' is an abbreviation for: 'For all x, $x = 2 \implies x^2 = 4$'. (Same with question 5.)

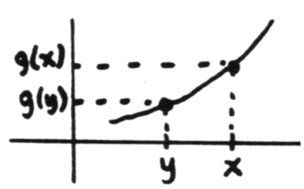
   At the bottom of page 290, the sentence:

   'IF f has a local extreme value at the point $(c, f(c))$, THEN the point $(c, f(c))$ is a critical point'

   is an abbreviation for: For all functions f and $c \in \mathcal{D}(f)$, IF f has ...

   In Exercise 6, question 2, the sentence 'If f has a local maximum at $x = 4$, then $f'(4) = 0$' is an abbreviation for 'For all functions f, if f has a local maximum at $x = 4$, then $f'(4) = 0$'. To prove that this (implied) 'for all' sentence is false, we exhibited a function for which the implication is false.

2. TRUE. This is a consequence of the DEFINITION of an increasing function.
3. TRUE. This is a consequence of the DEFINITION of a decreasing function.
4. FALSE. If the words '$x < y$' are inserted in the appropriate place, then it would be true. Here's a counterexample:



5. TRUE. This is a consequence of the DEFINITION of a decreasing function. Note that whenever the sentence '$g(y) < g(x)$' is true, then the sentence '$g(y) \leq g(x)$' is also true.
6. TRUE. This is a consequence of the DEFINITION of a nondecreasing function.
7. TRUE. This is a consequence of the DEFINITION of a nonincreasing function.

8. TRUE. This is a consequence of an important theorem.

9. FALSE. A function f can increase without being differentiable! The function graphed below is increasing on (a, b), but is not differentiable everywhere on (a, b).

10. TRUE. This states, precisely, that the only way that a product of two numbers can be positive is for both factors to be positive, or both factors to be negative.

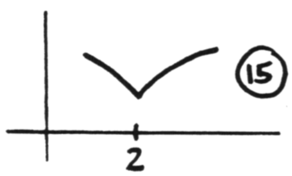
11. TRUE

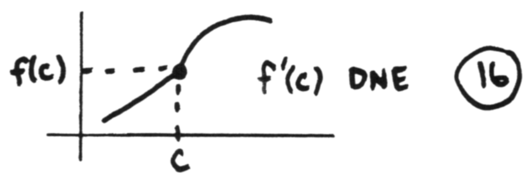
12. TRUE

13. TRUE. This is a consequence of the Intermediate Value Theorem.

14. TRUE. Every local extreme value MUST OCCUR at a critical point.

15. FALSE. The function f graphed below has a local minimum at $x = 2$, but $f'(2) \neq 0$.





16. FALSE. The point $(c, f(c))$ above is a critical point, but is not a local max or min.

17. TRUE. Since $\mathcal{D}(f') = \mathbb{R}$, $f'(c)$ must exist. Since $\mathcal{D}(f) = \mathbb{R}$, there are no endpoints of the domain of f.

18. FALSE. Take P to be false and Q to be true. Then, $P \Rightarrow Q$ is true, and $Q \Rightarrow P$ is false. Thus, the sentence '$P \Rightarrow Q$ is true' is true, but the sentence '$Q \Rightarrow P$ is true' is false.

19. TRUE. This is a consequence of the First Derivative Test.

20. TRUE. This is a consequence of a First Derivative Test at an endpoint.