CHAPTER 2. FUNCTIONS

Section 2.1 Functions and Function Notation

Quick Quiz:

1. In the graph shown, y is a function of x, because for every x, there is a unique y. That is, the graph passes a vertical line test.

   However, x is not a function of y. It is NOT true that for every y, there is a unique x. That is, the graph does NOT pass a horizontal line test.

2. $$x^2 - y + 1 = 0 \iff y = x^2 + 1$$

   For every value of x, there is a unique value of y. Thus, y is a function of x.

3. $$\begin{aligned} x^2 - y + 1 = 0 &\iff x^2 = y - 1 \\ &\iff x = \pm\sqrt{y-1} \end{aligned}$$

   For each allowable y-value, there are *two* x-values. Therefore, x is NOT a function of y.

4. Calling the function f: $f(x) = (\frac{x}{2} - 3)^2$

5. $g(-1) = 2(-1)^2 - 1 = 2 - 1 = 1$

   $g(x^2) = 2(x^2)^2 - 1 = 2x^4 - 1$

End-of-Section Exercises:

1. $f(0) = 0^3 - 1 = -1$

   $f(1) = 1^3 - 1 = 0$

   $f(-1) = (-1)^3 - 1 = -1 - 1 = -2$

   $f(t) = t^3 - 1$

   $f(f(2))$; first find $f(2)$: $f(2) = 2^3 - 1 = 7$; then, $f(f(2)) = f(7) = 7^3 - 1 = 342$

3. $f(-2) = |-2| = 2$

   $f(t) = |t| = \begin{cases} t & \text{if } t \ge 0 \\ -t & \text{if } t < 0 \end{cases}$

   $f(-t) = |-t| = |t|$

   $f(x^2) = |x^2| = |x|^2$

5. $h(-x) = \frac{1}{-x} = -\frac{1}{x}$

   $h(h(x)) = h(\frac{1}{x}) = \frac{1}{1/x} = x$

   $h(\frac{1}{x}) = \frac{1}{1/x} = x$

   $h(x + \Delta x) = \frac{1}{x+\Delta x}$

   $h(|x|) = \frac{1}{|x|} = \left|\frac{1}{x}\right|$

7. $h(1,1) = 1^2 + 1^2 - 1 = 1$

   $h(x,x) = x^2 + x^2 - 1 = 2x^2 - 1$

   $h(y,x) = y^2 + x^2 - 1 = h(x,y)$

   $h(x + \Delta x, y + \Delta y) = (x + \Delta x)^2 + (y + \Delta y)^2 - 1$

Section 2.2 Graphs of Functions

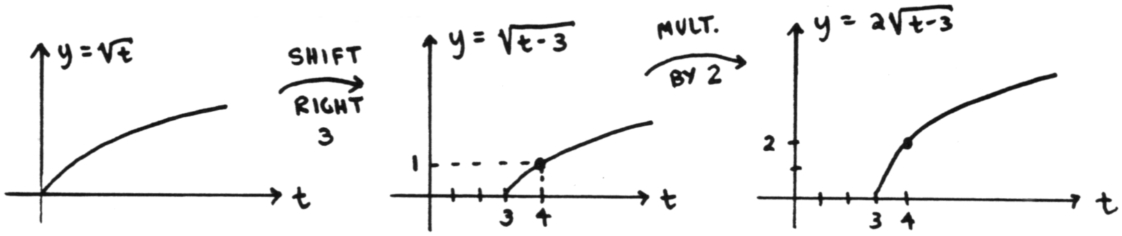
Quick Quiz:

1.

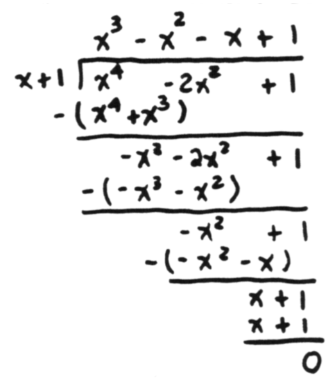
$$\begin{aligned}\mathcal{D}(f) &= \{x \mid 2x-1 \ge 0 \text{ and } x^2-9 \ne 0\}\\ &= \{x \mid 2x \ge 1 \text{ and } x^2 \ne 9\}\\ &= \{x \mid x \ge \frac{1}{2} \text{ and } |x| \ne 3\}\\ &= \{x \mid x \ge \frac{1}{2} \text{ and } x \ne 3 \text{ and } x \ne -3\}\\ &= \{x \mid x \ge \frac{1}{2} \text{ and } x \ne 3\}\end{aligned}$$

Note that if $x \ge \frac{1}{2}$, then automatically, x is not equal to -3.

2. TRUE. The order that elements are listed in a set is unimportant. In this sentence, the '=' sign is being used for equality of SETS.
3. By definition, the *graph of f* is the set of points $\{(x, f(x)) \mid x \in \mathcal{D}(f)\}$. More precisely, the graph usually refers to a (partial) *picture* of this set of points, in the xy-plane.
4. $\mathcal{D}(f) = [3, \infty)$; the graph is 'built up' below:



5. $P(-1) = (-1)^4 - 2(-1)^2 + 1 = 1 - 2 + 1 = 0$; therefore -1 is a root of P. Long division by $x - (-1) = x + 1$ yields:

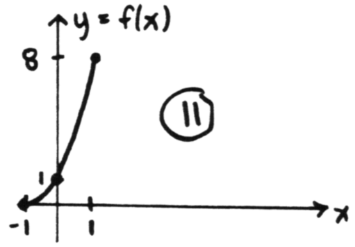


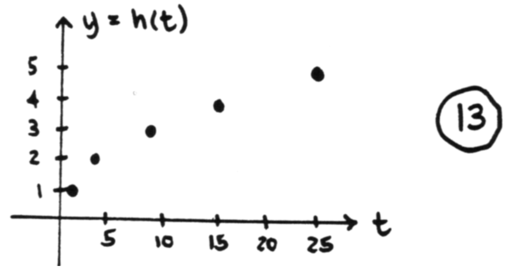
Therefore: $P(x) = (x+1)(x^3 - x^2 - x + 1)$

End-Of-Section Exercises:

1. EXP; this is a set.
3. SEN; TRUE. The component sentences '$x \ge 2$ and $x \ne 1$' and '$x \ge 2$' always have the same truth values. Both are true on $[2, \infty)$ and false elsewhere.
5. SEN; TRUE
7. SEN; TRUE. Both sets are equal to $\{3\}$.
9. SEN; this is TRUE (by definition), providing g is a function of one variable.

The graphs requested in problems 11 and 13 are given below:





Section 2.3 Composite Functions

Quick Quiz:

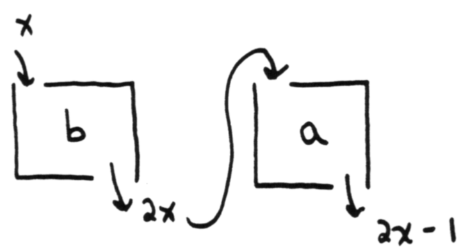
1. $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$. If $A = [1, 3)$ and $B = \{1, 2, 3\}$, then $A \cap B = \{1, 2\}$ since the only elements that are in *both* A and B are 1 and 2.
2. The sentence '$[1, 3] \subset \{1, 3\}$' is FALSE. For example, $2 \in [1, 3]$, but $2 \notin \{1, 3\}$.

   The sentence '$\{1, 3\} \subset [1, 3]$' is TRUE.

   The sentence 'For all sets A and B, $A \cap B \subset A$' is TRUE. Everything that is in BOTH A and B, is also in A.
3. $(f + g)(x) := f(x) + g(x)$

   $\mathcal{D}(f + g) = \{x \mid x \in \mathcal{D}(f) \text{ and } x \in \mathcal{D}(g)\} = \mathcal{D}(f) \cap \mathcal{D}(g)$
4. The function f takes an input x, multiplies by 2, then subtracts 1. Define $b(x) = 2x$ and $a(x) = x - 1$; then:

$$
\begin{aligned}
(a \circ b)(x) &:= a(b(x)) \\
&= a(2x) \\
&= 2x - 1 \\
&:= f(x)
\end{aligned}
$$



5. $\mathcal{R}(f) = \{1, -1\}$. The only two output values taken on by f are 1 and -1.

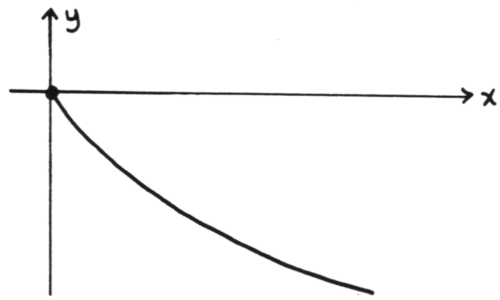
End-Of-Section Exercises:

1. EXP; $A \cup B$ is a set.
3. SEN; CONDITIONAL. The truth of the sentence '$A \subset B$' depends upon the sets A and B.
5. SEN; CONDITIONAL. The sentence '$\mathcal{R}(f) = \mathbb{R}$' states that the range of a function is the set of real numbers; the truth of this sentence depends upon the function f being referred to.
7. SEN; CONDITIONAL. The truth of this sentence depends upon the choice of functions f and g, and the choice of x.
9. SEN; FALSE. The set $\{a\}$ is NOT an element of the set $\{a, b\}$.
11. $\mathcal{R}(f) = [0, 8]$
13. $\mathcal{R}(h) = \{1, 2, 3, 4, 5\}$

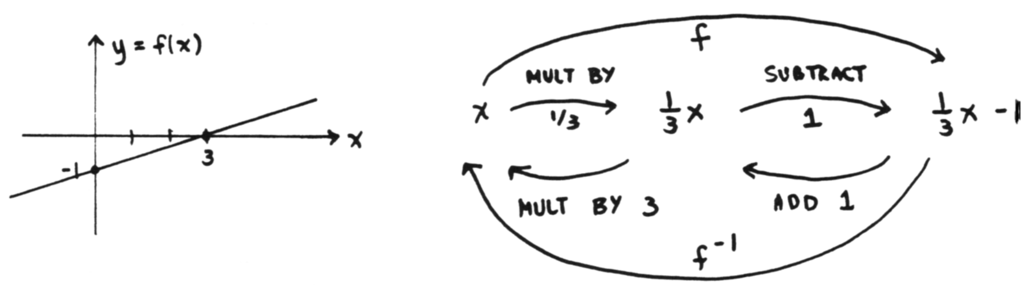
Section 2.4 One-to-One Functions and Inverse Functions

Quick Quiz:

1. The function $f(x) = x^2$ is NOT a one-to-one function. It does NOT have the property for every y, there is a unique x. That is, it does NOT pass the horizontal line test.
2. Translation: 'For every y in the range of f, there exists a unique x in the domain of f.' This is the 'one-to-one' condition for a function f.
3. One correct graph is shown below:

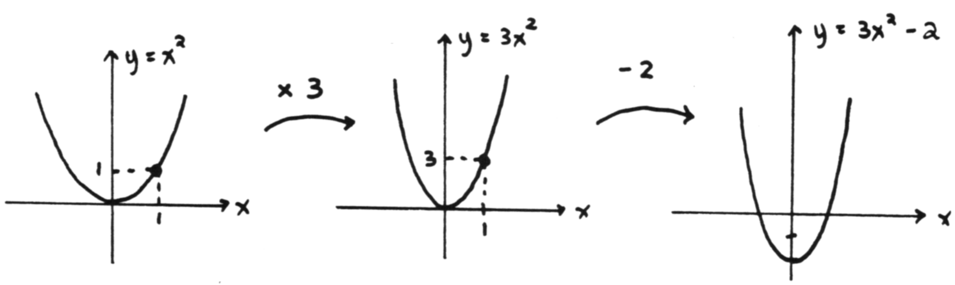


4. $f(f^{-1}(x)) = x \ \ \forall\, x \in \mathcal{R}(f)$
   $f^{-1}(f(x)) = x \ \ \forall\, x \in \mathcal{D}(f)$
5. The graph of f is the line shown below; it is clearly $1-1$. The function f takes an input x, multiplies by $\frac{1}{3}$, then subtracts 1; to 'undo' this, f^{-1} must take an input x, add 1, then divide by $\frac{1}{3}$ (that is, multiply by 3). Thus: $f^{-1}(x) = 3(x+1)$



End-Of-Section Exercises:

1. EXP; this is a function f^{-1}, evaluated at x
3. SEN; T
5. EXP
7. SEN; T
9. EXP
11. $\mathcal{D}(f) = \mathbb{R}$, $\mathcal{R}(f) = (-2, \infty)$



13. $\mathcal{D}(h) = \mathbb{R}$, $\mathcal{R}(h) = (5, \infty)$