## 6.2 Some Basic Antidifferentiation Formulas

*every differentiation formula has a 'counterpart' antidifferentiation formula*

Every differentiation formula has a 'counterpart' antidifferentiation formula. For example:

$$\frac{d}{dx}(x) = 1 \quad \text{has 'counterpart'} \quad \int (1)\,dx = x + C$$

Why? The statement $\frac{d}{dx}(x) = 1$ tells us that $x$ is an antiderivative of $1$. That is, $x$ is a function which, when differentiated, yields $1$. Then, all other antiderivatives must have precisely the same shape; they can differ by at most a constant.

Similarly:

$$\frac{d}{dx}(x^2) = 2x \quad \text{has 'counterpart'} \quad \int 2x\,dx = x^2 + C$$

In this latter case, it would be more useful to have a formula for $\int x\,dx$, instead of $\int 2x\,dx$. Using the linearity of the integral, this is easy to get:

$$\begin{aligned} \int 2x\,dx = x^2 + C \quad &\iff \quad 2\int x\,dx = x^2 + C && \text{(linearity)} \\ &\iff \quad \int x\,dx = \frac{(x^2 + C)}{2} && \text{(divide by 2)} \\ &\iff \quad \int x\,dx = \frac{x^2}{2} + K && \text{(rewrite constant)} \end{aligned}$$

Since $C$ is an arbitrary constant, so is $\frac{C}{2}$. There is no sense in giving an arbitrary constant a complicated name like $\frac{C}{2}$; so change the name to, say, $K$.

Thus, we have learned that:

$$\int x\,dx = \frac{x^2}{2} + C$$

*using the formula* $\int x\,dx = \frac{x^2}{2} + C$

With the formula

$$\int x\,dx = \frac{x^2}{2} + C$$

in hand, and linearity of the integral, a number of integration problems can be easily solved.

**EXAMPLE**

Problem: Evaluate $\int 3x\,dx$ .

Solution: First, the solution is written in strictly correct, painstaking detail. Then, it is shown how the solution is commonly abbreviated.

$$\begin{aligned}
\int 3x\,dx &= 3\int x\,dx && \text{(linearity)} \\
&= 3(\frac{x^2}{2}+C) && \text{(use formula for integrating } x\text{)} \\
&= \frac{3x^2}{2}+3C && \text{(multiply)} \\
&= \frac{3x^2}{2}+K && \text{(rewrite constant)}
\end{aligned}$$

In practice, one recognizes that the final result will always have an added arbitrary constant. So: simply apply the formulas *without* the arbitrary constant, and in the final step, remember to include it. This yields the common solution appearance:

$$\int 3x\,dx = 3\int x\,dx = 3(\frac{x^2}{2}) + C = \frac{3x^2}{2} + C$$

Similarly, one writes

$$\int (\pi t - 4)\,dt = \pi\frac{t^2}{2} - 4t + C$$

and:

$$\int \frac{2-t}{7}\,dt = \frac{1}{7}(2t - \frac{t^2}{2}) + C = \frac{4t-t^2}{14} + C$$

**EXERCISE 1**

♣ 1. What is the antidifferentiation 'counterpart' to the differentiation formula

$$\frac{d}{dx}(x^3) = 3x^2 \ ?$$

♣ 2. Use your 'counterpart' to obtain a formula for $\int x^2\,dx$ .

♣ 3. Use your formula for integrating $x^2$ to evaluate $\int 5x^2\,dx$ .

The next integration formula derives from the Simple Power Rule for Differentiation:

$$\frac{d}{dx}x^n = nx^{n-1}$$

It is thus appropriately named the '*Simple Power Rule for Integration*'.

*Simple Power Rule for Integration*

Let $n$ be any number except $-1$. Then:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

This formula is referred to as the *Simple Power Rule for Integration.*

To verify this result, one need only check that:

$$\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = \frac{1}{n+1}\frac{d}{dx}(x^{n+1}) = \frac{1}{n+1}(n+1)x^{(n+1)-1} = x^n$$

Together with algebraic manipulation and linearity of the integral, this formula allows us to solve a wide variety of antidifferentiation problems, as the following examples illustrate.

**EXAMPLE**

Problem: Evaluate $\int x^{-3}\,dx$ .

Solution:

$$\int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

Check: $\frac{d}{dx}\left(-\frac{1}{2x^2}\right) = \frac{d}{dx}(-\frac{1}{2}x^{-2}) = -\frac{1}{2}(-2)x^{-3} = x^{-3}$

**EXAMPLE**

Sometimes it is necessary to *rewrite* the integrand before integrating:

Problem: Evaluate $\int \frac{1}{x^2}\,dx$ .

Solution:

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

Check: $\frac{d}{dx}(-\frac{1}{x}) = \frac{d}{dx}(-x^{-1}) = -(-1)x^{-1-1} = x^{-2} = \frac{1}{x^2}$

Here, it was necessary to get the integrand into a form that could be handled by the Simple Power Rule for Integration.

**EXAMPLE**

Problem: Evaluate $\int t(t^2+1)\,dt$ .

Solution:

$$\int t(t^2+1)\,dt = \int (t^3+t)\,dt = \frac{t^4}{4} + \frac{t^2}{2} + C$$

Check!

♣ Where was linearity of the integral used here?

**EXAMPLE**

Problem: Evaluate $\int (\sqrt[4]{x^3}+1)\,dx$ .

Solution:

$$\begin{aligned}
\int (\sqrt[4]{x^3}+1)\,dx &= \int (x^{3/4}+1)\,dx && \text{(rewrite)}\\
&= \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} + x + C && \text{(use formulas and linearity)}\\
&= \frac{4}{7}x^{7/4} + x + C && (\frac{3}{4}+1 = \frac{3}{4}+\frac{4}{4} = \frac{7}{4})\\
&= \frac{4}{7}(x^7)^{\frac{1}{4}} + x + C && ((x^a)^b = x^{ab})\\
&= \frac{4}{7}\sqrt[4]{x^7} + x + C
\end{aligned}$$

It's a good rule of thumb to get your final answer in a form that matches, as closely as possible, the original form of the problem. Since the original problem was given in radical form (not fractional exponent form), the final answer was also given in radical form.

**EXAMPLE**

Problem: Evaluate $\int \frac{x^2+1}{x^2}\,dx$.
Solution:

$$\begin{aligned}\int \frac{x^2+1}{x^2}\,dx &= \int 1 + \frac{1}{x^2}\,dx \\ &= \int 1 + x^{-2}\,dx \\ &= x + \frac{x^{-1}}{-1} + C \\ &= x - \frac{1}{x} + C \\ &= \frac{x^2-1}{x} + C\end{aligned}$$

**EXAMPLE**

Problem: Evaluate $\int (3y^2-1)^2\,dy$.
Solution:

$$\begin{aligned}\int (3y^2-1)^2\,dy &= \int (9y^4 - 6y^2 + 1)\,dy \\ &= 9(\frac{y^5}{5}) - 6(\frac{y^3}{3}) + y + C \\ &= \frac{9}{5}y^5 - 2y^3 + y + C\end{aligned}$$

**EXERCISE 2**

Evaluate the following integrals. Be sure to write complete mathematical sentences. Don't forget to include the arbitrary constant. Check your answers.

♣ 1. $\displaystyle\int (ax^2 + bx + c)\,dx$, where $a$, $b$ and $c$ are constants

♣ 2. $\displaystyle\int \frac{2\sqrt{t}-1}{t^2}\,dt$

♣ 3. $\displaystyle\int (1+\sqrt[3]{x})^2\,dx$

♣ 4. $\displaystyle\int \sqrt{\frac{3\pi}{y^4}} - e^y\,dy$

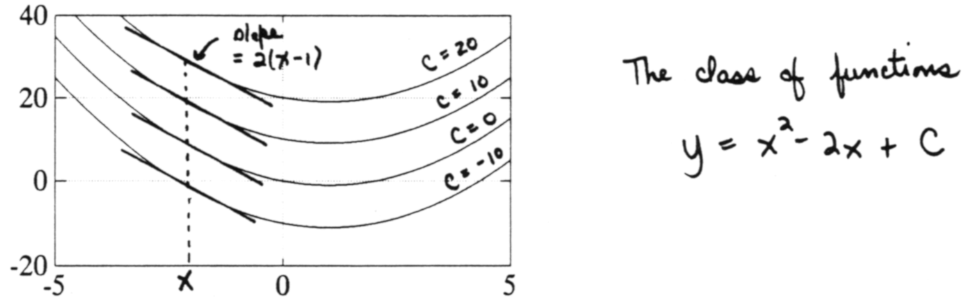
♣ 5. $\displaystyle\int \left(\frac{\sqrt{x}-1}{x}\right)^2 dx$

*finding a particular solution*

An integration problem like

$$\int 2(x-1)\,dx = 2(\frac{x^2}{2} - x) + C = x^2 - 2x + C$$

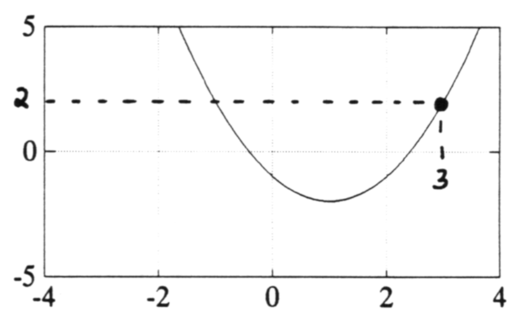
yields a whole *class* of functions, each of which has derivative $2(x-1)$. Some members of this class are shown below:




Occasionally, it is desired to go into this class, and choose a *particular* member; one that passes through a specified point. For example, if we want a function $f$ satisfying the two properties

- $f'(x) = 2(x-1)$, and
- $(3,2)$ lies on the graph of $f$

then we must find the constant $C$ corresponding to the function shown below:




When will the function $f(x) = x^2 - 2x + C$ pass through the point $(3,2)$? Precisely when $f(3) = 2$:

$$\begin{aligned} f(3) = 2 \quad &\iff \quad 3^2 - 2(3) + C = 2 \\ &\iff \quad 3 + C = 2 \\ &\iff \quad C = -1 \end{aligned}$$

Thus, the desired function is $f(x) = x^2 - 2x - 1$. Problems such as this are called '*Finding a particular solution*'.

**EXAMPLE**

Problem: Find a function $y$ satisfying:

- $\frac{dy}{dx} = x^2 + 2$, and
- the point $(1,5)$ lies on the graph of $y$

First, find *all* functions $y$ with derivative $x^2+2$:

$$y = \int (x^2+2)\,dx = \frac{x^3}{3} + 2x + C$$

Since the desired curve is to contain the point $(1,5)$, $C$ must be chosen to satisfy the property that $y = 5$ when $x = 1$:

$$\begin{aligned} (1,5) \text{ on curve} \quad &\iff \quad 5 = \frac{1^3}{3} + 2(1) + C \\ &\iff \quad C = 3 - \frac{1}{3} = \frac{8}{3} \end{aligned}$$

Thus, $y = \frac{x^3}{3} + 2x + \frac{8}{3}$ is the desired curve.

**EXERCISE 3**

*finding particular solutions*

♣ 1. Find a function $y$ with derivative $2x-3$, that passes through the point $(0,4)$.

♣ 2. Find a function $f$ satisfying the following properties:

a) $f'(x) = \sqrt{x}$, and

b) $f(1) = -2$

**EXERCISE 4**

♣ 1. Find a function $f$ satisfying all the following properties:

a) $f'(x) = 2$ for $x > 1$

b) $f'(x) = 3x^2$ for $x < 1$

c) $f(1) = 0$

d) $f$ is continuous at $x = 1$

♣ 2. Find another function $f$ satisfying all the properties above except the last: this time, $f$ should have a nonremovable discontinuity at $x = 1$.

*integrating $e^{kx}$*

The antidifferentiation 'counterpart' of the differentiation formula $\frac{d}{dx}(e^{kx}) = ke^{kx}$ is:

$$\int ke^{kx}\,dx = e^{kx} + K \quad \iff \quad \int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

Summarizing:

*integrating $e^{kx}$*

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

**EXAMPLE**

Problem: Evaluate $\int e^{3x}\,dx$.

Solution:

$$\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$$

**EXAMPLE**

Problem: Evaluate $\int e^{2t-1}\,dt$.

Solution:

$$\begin{aligned}\int e^{2t-1}\,dt &= \int e^{2t}e^{-1}\,dt \\ &= e^{-1}\int e^{2t}\,dt \\ &= e^{-1}(\frac{1}{2}e^{2t}) + C \\ &= \frac{1}{2}e^{2t-1} + C\end{aligned}$$

Check: $\frac{d}{dt}(\frac{1}{2}e^{2t-1}) = \frac{1}{2}(e^{2t-1})(2) = e^{2t-1}$

*integrating* $x^{-1} = \frac{1}{x}$

Note that when $n = -1$, the Simple Power Rule for Integration does not apply, because the formula $\frac{x^{n+1}}{n+1}$ is not defined. Therefore, this rule cannot be used to tell us how to integrate $\int x^{-1}\,dx = \int \frac{1}{x}\,dx$.

However, we *do* know a function whose derivative is $\frac{1}{x}$:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Thus:

$$\int \frac{1}{x}\,dx = \ln x + C$$

However, there's something undesirable about this formula. The function $\frac{1}{x}$ is defined for all $x$ except 0; however the antiderivatives $\ln x + C$ are only defined for positive $x$. This problem can be remedied, and is the next topic of discussion.

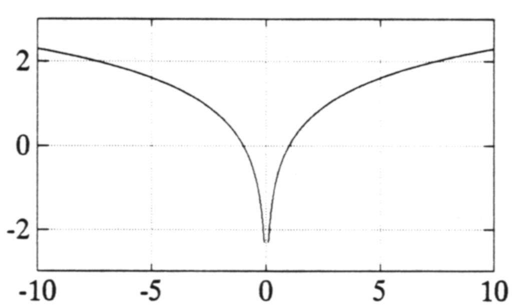
*investigating* $\frac{d}{dx}\ln|x|$

The function $y = \ln|x|$ has the graph shown below. Note that:

$$\ln|x| = \begin{cases} \ln x & \text{for } x > 0 \\ \ln(-x) & \text{for } x < 0 \end{cases}$$

The domain of $\ln|x|$ is precisely the same as the domain of $\frac{1}{x}$: all nonzero $x$.

Now, is $\ln|x|$ an antiderivative of $\frac{1}{x}$? That is, does $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$? It is shown next that the answer is 'Yes'!




GRAPH OF
$y = \ln|x|$

Each 'piece' of the function is differentiated separately.
For $x > 0$: $\frac{d}{dx} \ln x = \frac{1}{x}$
For $x < 0$: $\frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$
In either case the same formula is obtained, so that for all $x \neq 0$:

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

The antiderivative $\ln |x|$ should always be used when integrating $\frac{1}{x}$.

*integrating $\frac{1}{x}$*

$$\int \frac{1}{x}\, dx = \ln |x| + C$$

**EXAMPLE**

Problem: Find all the antiderivatives of $\frac{1}{3x}$ .
Solution:

$$\begin{aligned} \int \frac{1}{3x}\, dx &= \frac{1}{3} \int \frac{1}{x}\, dx \\ &= \frac{1}{3} \ln |x| + C \end{aligned}$$

**EXERCISE 5**

Find all the antiderivatives of the following functions. Be sure to write your answers using complete mathematical sentences.

♣ 1. $f(x) = \frac{1-\sqrt{x}}{x}$

♣ 2. $y = \left(\frac{t+1}{t}\right)^2$

♣ 3. $g(x) = \frac{1}{7x} + e^{-x} + 1$

**QUICK QUIZ**
*sample questions*

1. What is the antidifferentiation 'counterpart' to the differentiation formula $\frac{d}{dx} e^{kx} = ke^{kx}$ ?
2. Find: $\int \sqrt{x}\, dx$
3. Find: $\int \frac{1}{2t}\, dt$
4. Find a function $f$ satisfying: $f'(x) = x$ and $f(0) = 3$

**KEYWORDS**
*for this section*

*Differentiation 'counterparts', Simple Power Rule for Integration, finding particular solutions, integrating $e^{kx}$, integrating $\frac{1}{x}$.*

**END-OF-SECTION EXERCISES**

♣ Write three antidifferentiation problems, that can be solved with the tools available to you.

The first problem should involve a radical; the second a binomial squared, and the third a rational function.

Solve the three antidifferentiation problems, and then check, by differentiating.