5.6 Asymptotes; Checking Behavior at Infinity

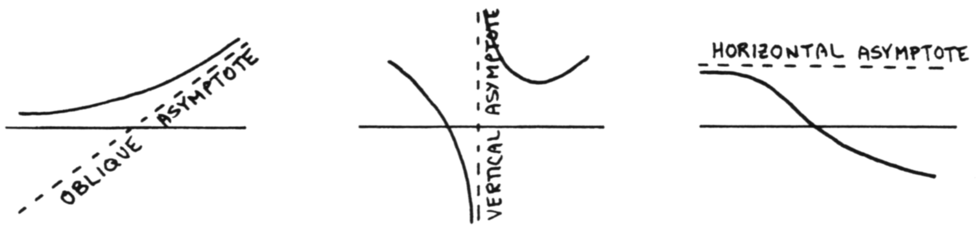
checking behavior at infinity

In this section, the notion of *checking behavior at infinity* is made precise, by discussing both *asymptotes* and *limits involving infinity.*

DEFINITION

asymptote

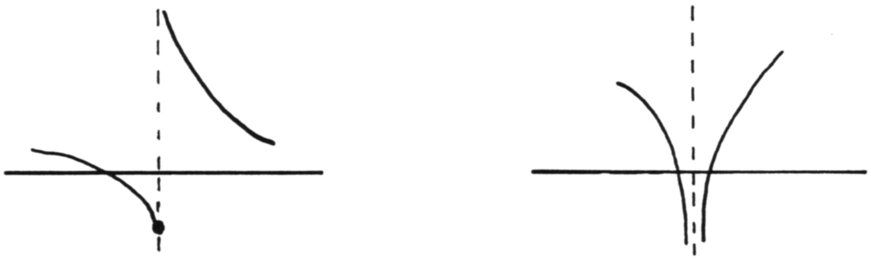
An *asymptote* is a curve (usually a line) that a graph gets arbitrarily close to as x approaches $\pm\infty$, or as x approaches some finite number.



vertical asymptotes

An asymptote that is a vertical line is called a *vertical asymptote.*

That is, if the numbers $f(x)$ approach $\pm\infty$ as x approaches c from the right or left (or both), then the line $x = c$ is a vertical asymptote for f.



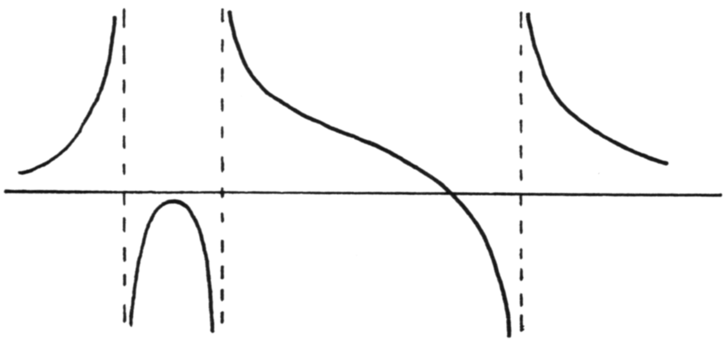
How do vertical asymptotes arise?

Vertical asymptotes arise most naturally when dealing with rational functions (ratios of polynomials):

$$f(x) = \frac{N(x)}{D(x)}$$

Any value of x for which the denominator is zero (and the numerator is nonzero) gives rise to a vertical asymptote.

A function can have an unlimited number of vertical asymptotes.



DEFINITION $\lim_{x\to c^+} f(x) = \infty$

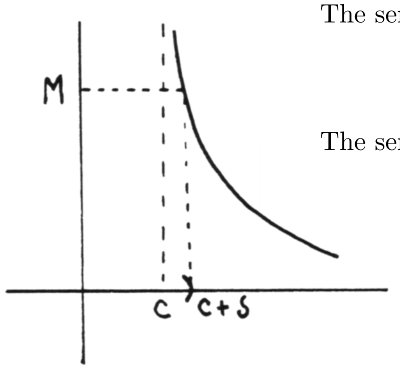
The limit statement

$$\lim_{x\to c^+} f(x) = \infty$$

means that $f(x)$ can be made as large and positive as desired, by requiring that x be sufficiently close to c (and greater than c).

Precisely:

$$\lim_{x\to c^+} f(x) = \infty \quad \Longleftrightarrow \quad \begin{array}{l}\forall\ M > 0,\ \exists\ \delta > 0 \text{ such that if} \\ x \in (c, c+\delta), \text{ then } f(x) > M\end{array}$$



The sentence $\lim_{x\to c^+} f(x) = \infty$ can also be written:

$$\text{As } \ x \to c^+,\ \ f(x) \to \infty$$

The sentence '$\lim_{x\to c^+} f(x) = \infty$' is read as:

the limit of $f(x)$,
as x approaches c from the right-hand side,
is infinity

EXERCISE 1

♣ Give a precise definition of:

$$\lim_{x\to c^-} f(x) = \infty$$

Make a sketch that illustrates this limit statement. In English, what is this definition saying?

EXERCISE 2

♣ Give a precise definition of:

$$\lim_{x\to c^+} f(x) = -\infty$$

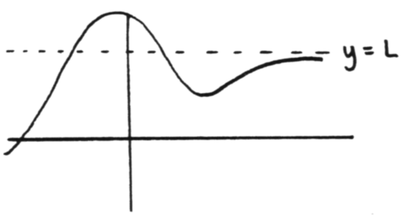
Make a sketch that illustrates this limit statement. In English, what is this definition saying?

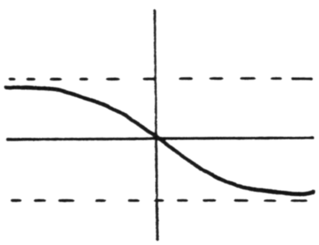
horizontal asymptotes

An asymptote that is a horizontal line is called a *horizontal asymptote.*

That is, if the line $y = L$ is a horizontal asymptote for f, then the function values $f(x)$ approach the finite number L as x approaches $+\infty$ or $-\infty$ (or both).

A *function* can have at most two horizontal asymptotes. (♣ Why?)





How do horizontal asymptotes arise?

Horizontal asymptotes also arise most naturally when dealing with rational functions,

$$f(x) = \frac{P(x)}{D(x)} \ ,$$

when the degrees of the numerator and denominator are the same.

For example, consider:

$$f(x) = \frac{3x^2 - 1}{x^2 - 2x + 2}$$

To investigate the behavior of f for large values of x, first multiply by 1 in an appropriate form (the highest power of x that appears, over itself):

$$\begin{aligned} f(x) &= \frac{3x^2 - 1}{x^2 - 2x + 2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \frac{3 - \frac{1}{x^2}}{1 - \frac{2}{x} + \frac{2}{x^2}} \end{aligned}$$

In this form, it is easy to see that when x is large (positive or negative), $f(x)$ is close to 3. Precisely, recall that the limit of a quotient is the quotient of the limits, *provided that each individual limit exists.* Since both 'numerator' and 'denominator' limits exist:

$$\lim_{x\to\pm\infty}(3 - \frac{1}{x^2}) = 3 \quad \text{and} \quad \lim_{x\to\pm\infty}(1 - \frac{2}{x} + \frac{2}{x^2}) = 1 \ ,$$

it is correct to say that:

$$\lim_{x\to\pm\infty} \frac{3 - \frac{1}{x^2}}{1 - \frac{2}{x} + \frac{2}{x^2}} = \frac{\lim_{x\to\pm\infty}(3 - \frac{1}{x^2})}{\lim_{x\to\pm\infty}(1 - \frac{2}{x} + \frac{2}{x^2})} = \frac{3}{1} = 3$$

Thus, the line $y = 3$ is a horizontal asymptote for the graph of f.

abbreviated form

Instead of writing out all the steps indicated above, the author usually summarizes things as follows:

Problem: Investigate the behavior of $f(x) = \frac{3x^2-1}{x^2-2x+2}$ for large values of x.

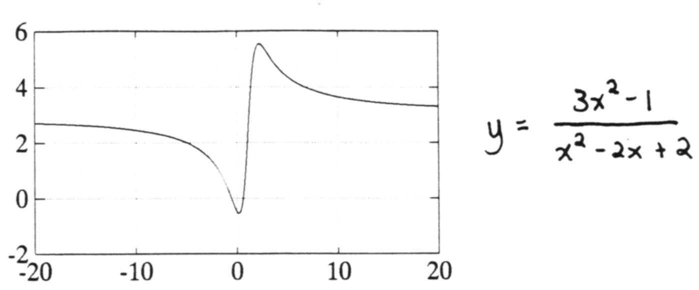
Solution: Think of approximating both the numerator and denominator polynomials by their highest order terms, and simply write:

$$\text{For large } x, \ \ f(x) \approx \frac{3x^2}{x^2} = 3$$

Thus, $y = 3$ is a horizontal asymptote for f.

This abbreviated analysis is fine, *provided that you understand why it is justified, and can fill in the details if pressed to do so.*

A MATLAB graph of f is shown below. Observe that there are no real numbers x for which the denominator $x^2 - 2x + 2$ equals zero, so f has no vertical asymptotes.



EXERCISE 3

♣ Investigate

$$f(x) = \frac{5x^3}{2x(x-1)(x+1)}$$

for horizontal asymptote behavior.

Write both a precise solution, and an abbreviated solution.

DEFINITION

$\lim_{x\to\infty} f(x) = L$

The limit statement

$$\lim_{x\to\infty} f(x) = L$$

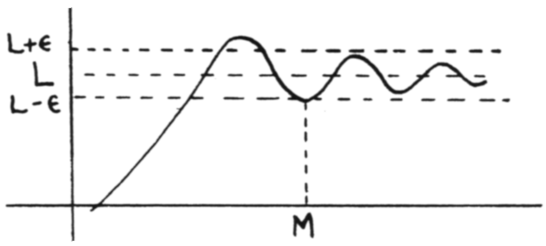
means that the numbers $f(x)$ can be made as close to L as desired, by requiring that x be sufficiently large and positive.

Precisely:

$$\lim_{x\to\infty} f(x) = L \iff \begin{array}{l} \forall\ \epsilon > 0,\ \exists\ M > 0 \text{ such that if } x > M, \\ \text{then } |f(x) - L| < \epsilon \end{array}$$

The sentence $\lim_{x\to\infty} f(x) = L$ can also be written:

$$\text{As } \ x \to \infty,\ \ f(x) \to L$$



EXERCISE 4 ♣ Give a precise definition of:

$$\lim_{x\to -\infty} f(x) = L$$

Make a sketch that illustrates this limit statement. In English, what is this definition saying?

oblique asymptotes

An asymptote that is a line, but not a vertical or horizontal line, is called an *oblique asymptote*.

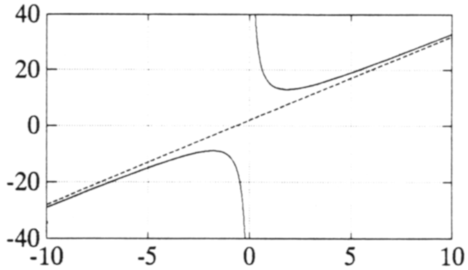
For example, consider the function:

$$f(x) = \frac{3x^2 + 2x + 10}{x} = 3x + 2 + \frac{10}{x}$$

For large values of x (positive or negative), the number $\frac{10}{x}$ is close to zero. Thus, for large values of x,

$$f(x) \approx 3x + 2 \ ,$$

and the line $y = 3x + 2$ is an oblique asymptote for f. The graph of f is shown below, along with the line $y = 3x + 2$.



Caution!

Do not 'abuse' the abbreviated solution technique! It is fine to say: for large values of x, $f(x) \approx \frac{3x^2}{x} = 3x$, and from this gain the information that when x is large, so is $f(x)$. However, it is *not* correct to infer that $y = 3x$ is an oblique asymptote! Observe that the 'multiply by 1 in an appropriate form' technique breaks down for this example:

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{3x^2+2x+10}{x} &= \lim_{x\to\pm\infty} \frac{3x^2+2x+10}{x} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \lim_{x\to\pm\infty} \frac{3 + \frac{2}{x} + \frac{10}{x^2}}{\frac{1}{x}} \ ,\end{aligned}$$

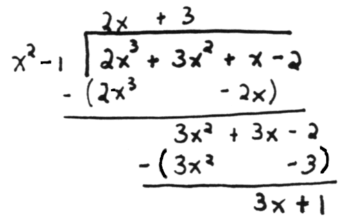
but the limit of the quotient *cannot* be written as the quotient of the limits, since the denominator tends to 0.

When the degree of the numerator is greater than the degree of the denominator, the correct technique is to rewrite the rational function as a sum, by doing a long division. This is illustrated in the next example.

EXAMPLE

Problem: Find the oblique asymptote for $f(x) = \frac{2x^3+3x^2+x-2}{x^2-1}$.

Solution: As x gets large, so does $f(x)$. Do a long division:



Remember to stop when the degree of the remainder is strictly less than the degree of the divisor. Thus:

$$f(x) = 2x + 3 + \frac{3x+1}{x^2-1}$$

As $x \to \pm\infty$, $\frac{3x+1}{x^2-1} \to 0$. Thus, when x is large,

$$f(x) \approx 2x + 3 \ ,$$

and the line $y = 2x + 3$ is an oblique asymptote for f.

EXAMPLE

graphing a rational function

Problem: Completely graph $f(x) = \frac{x}{x-1}$.

Solution:

- $\mathcal{D}(f) = \{x \mid x \neq 1\}$
  Plot a few points:

x	$f(x)$
0	0
2	2
-2	2/3
-1	1/2

Check behavior near $x = 1$:

As $x \to 1^+$, $f(x) \to +\infty$. A convenient way to check this and write it down is:

$$f(1^+) \approx \frac{(+)}{(\text{small } +)} \to +\infty$$

The notation $f(1^+)$ connotes that f is being investigated on numbers that are a little bit greater than 1; say, 1.01 and 1.001 .

The notation $\frac{(+)}{(\text{small } +)}$ connotes a positive number divided by a small positive number, which yields a large positive number. For example: $\frac{1.01}{(1.01-1)} = \frac{1.01}{0.01} = 101$

Also:

$$f(1^-) \approx \frac{(+)}{(\text{small } -)} \to -\infty$$

That is, as $x \to 1^-$, $f(x) \to -\infty$.

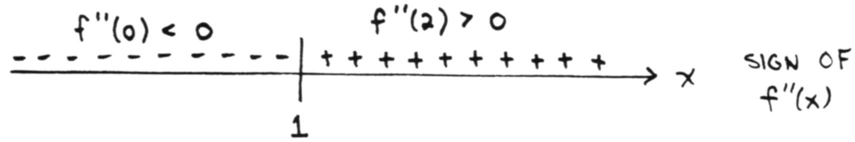
For example: $\frac{0.99}{0.99-1} = \frac{0.99}{-0.01} = -99$

Thus, $x = 1$ is a vertical asymptote.

- Compute the first derivative: $f'(x) = \frac{(x-1)(1)-x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$

  $\mathcal{D}(f') = \mathcal{D}(f)$, and $f'(x)$ never equals 0. There are no critical points.
- Compute the second derivative: $f''(x) = \frac{2}{(x-1)^3}$

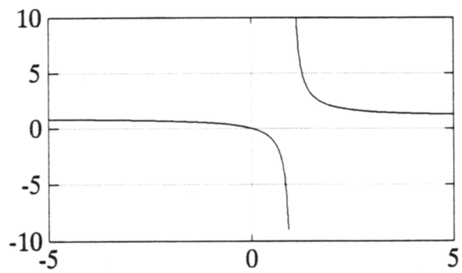
  $\mathcal{D}(f'') = \mathcal{D}(f)$, and $f''(x)$ never equals 0. There are no candidates for inflection points.
- Sign of the second derivative:



- Filling in some details:

  As $x \to \pm\infty$, $f(x) \approx \frac{x}{x} = 1$, so $y = 1$ is a horizontal asymptote.

A MATLAB graph of $f(x) = \frac{x}{x-1}$ is shown below.



EXAMPLE

graphing a rational function

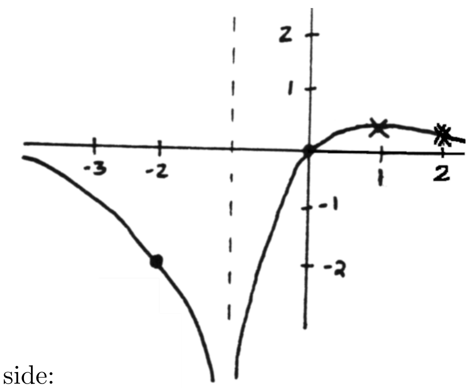
Problem: Completely graph $f(x) = \frac{x}{(x+1)^2}$.

Solution:

- $\mathcal{D}(f) = \{x \,|\, x \neq -1\}$

  Plot a few points:

x	f(x)
0	0
1	1/4
-2	-2
2	2/9



Check behavior near $x = -1$:

First, coming in to -1 from the right-hand side:

$$f(-1^+) \approx \frac{(-)}{(\text{small } +)} \to -\infty$$

Thus, as $x \to -1^+$, $f(x) \to -\infty$.

Next, coming in to -1 from the left-hand side:

$$f(-1^-) \approx \frac{(-)}{(\text{small } +)} \to -\infty$$

So, as $x \to -1^-$, $f(x) \to -\infty$.

The line $x = -1$ is a vertical asymptote.

- Compute the first derivative:

$$\begin{aligned} f'(x) &= \frac{(x+1)^2(1) - x\cdot 2(x+1)}{(x+1)^4} \\ &= \frac{(x+1)(x+1-2x)}{(x+1)^4} \\ &= \frac{1-x}{(x+1)^3} \end{aligned}$$

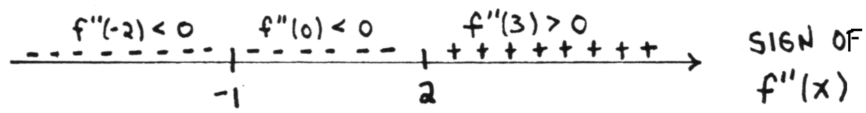
  $\mathcal{D}(f') = \mathcal{D}(f)$; $f'(x) = 0$ when $x = 1$, so $(1, f(1)) = (1, \frac{1}{4})$ is a critical point. Plot this point with a $\times$.

- Compute the second derivative:

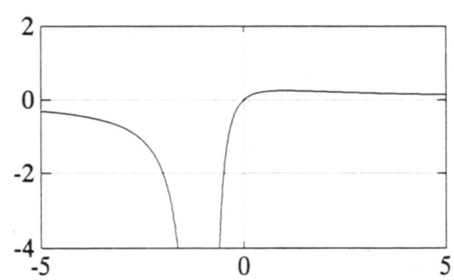
$$\begin{aligned} f''(x) &= \frac{(x+1)^3(-1) - (1-x)3(1+x)^2}{(x+1)^6} \\ &= \frac{(x+1)^2[-(x+1) - 3(1-x)]}{(1+x)^6} \\ &= \frac{2x-4}{(1+x)^4} \end{aligned}$$

  $\mathcal{D}(f'') = \mathcal{D}(f)$; $f''(x) = 0$ when $x = 2$. Thus, $(2, f(2)) = (2, \frac{2}{9})$ is a possible inflection point. Plot this point with a ⋇ .

- Sign of f'' :



A MATLAB graph of $f(x) = \frac{x}{(x+1)^2}$ appears below.



- Fill in some details:
  As $x \to \pm\infty$, $f(x) \approx \frac{x}{x^2} = \frac{1}{x} \to 0$, so the line $y = 0$ is a horizontal asymptote.

- Read off important information:
  $(1, \frac{1}{4})$ is a local maximum
  $(2, \frac{2}{9})$ is an inflection point
  The line $y = 0$ is a horizontal asymptote.
  The line $x = -1$ is a vertical asymptote.
  The graph is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1) \cup (1, \infty)$.
  The graph is concave down on $(-\infty, -1) \cup (-1, 2)$ and concave up on $(2, \infty)$.

EXERCISE 5

Completely graph each of the following functions. Be sure to check for horizontal, vertical, and oblique asymptotes.

♣ 1. $f(x) = \dfrac{2x^3 - x^2 + 1}{x^2}$

♣ 2. $g(x) = \dfrac{x^2 - 3}{x^2 - 1}$

♣ 3. $y = \dfrac{1 - 4x^2}{x^2 + 1}$

EXERCISE 6

♣ Completely graph: $f(x) = \dfrac{x^3 + 2x^2 - x - 2}{x^2 - 1}$ (Be careful!)

QUICK QUIZ
sample questions

1. What is an *asymptote*?
2. Write down a *precise* definition for the limit statement: $\lim_{x \to c^-} f(x) = -\infty$
3. Find all asymptotes (horizontal, vertical, oblique) for: $f(x) = \dfrac{3x - 1}{x + 2}$
4. Under what condition(s) is the limit of a quotient equal to the quotient of the limit?

KEYWORDS
for this section

Asymptotes, vertical, horizontal and oblique asymptotes, precise definitions of limits involving infinity.

END-OF-SECTION EXERCISES

♣ Re-do each of the graphing examples in this section, *without looking at the text.* Be sure to write complete mathematical sentences. If you get stuck, then study the text example, close the book, and try it yourself again.