3.3 Properties of Limits

This section establishes some useful properties of limits, the development of which provides additional practice with the concept of the limit of a function.

existence versus uniqueness

Mathematicians are extremely fond of *existence* and *uniqueness arguments.* An *existence argument* shows that a certain object *exists*, but does not address the issue: How many? A *uniqueness argument* answers the question 'How many?' with a definitive: Exactly one.

existence of $\lim_{x\to c} f(x)$

When does the limit $\lim_{x\to c} f(x)$ exist? The definition answers this question: it exists when there is a number l with the property that one can get $f(x)$ as close to l as desired, by requiring that $x \in \mathcal{D}(f)$ be sufficiently close to c, but not equal to c.

When $lim_{x\to c} f(x)$ *exists, is it unique?*

Is it possible that there are two different numbers l and k, *both* satisfying the definition of the limit of a function? Or, is the limit *unique*? If you stop to think about this for a moment, you'll probably conclude that $f(x)$ can't be close to *two different numbers* at the same time. But how can this be argued precisely?

the way mathematicians show uniqueness

The way mathematicians usually establish uniqueness is to:

- Suppose that there are *two*;
- Show that these two are the same.

a typical uniqueness argument

That is, suppose a mathematician is asked to prove the following theorem. (Remember, a *theorem* is a mathematical result that is both *important* and *true*.)

Theorem. An object with property P is unique.

Don't worry about what property P is; here we are discussing the *form* of a typical uniqueness argument, and are not concerned with specific *content*.

Here's how the proof would go:

Proof. Suppose that x and y both satisfy property P. (More stuff here.) Then, $x = y$. ∎

Early on in the proof, x could potentially be different from y; all that is known is that they both satisfy 'property P'. But then, information about 'property P' is used to show that x must equal y.

the symbol ∎ *is used to mark the end of proofs*

The symbol ∎ is an end-of-proof marker. It is really just a courtesy to the reader; a gentle reminder that the author has finished showing whatever was set out to be shown.

EXERCISE 1

♣ Prove that there is a unique solution to the linear equation

$$ax + b = c, \quad a \neq 0 \ ,$$

by supposing that both X and Y are solutions, and showing that $X = Y$. Be sure to write down complete mathematical sentences.

The next theorem states, in the language of mathematics, that limits are unique.

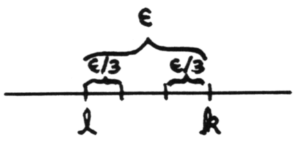
THEOREM
limits are unique

Suppose that:

$$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} f(x) = k$$

Then, $l = k$.

a motivation for the proof



Before jumping into the rigorous proof, just stop and think. How could it be shown that l must equal k?

If l is not equal to k, then there's some positive distance between them; call it ϵ. Since ϵ is positive, so is $\epsilon/3$. Looking back at the precise definition of the limit of a function, one observes that ϵ represents *any* positive number. The definition can certainly be applied, taking this positive number to be $\epsilon/3$. (If this seems awkward to you, rewrite the definition, using ω instead of ϵ. Then, take ω to be $\epsilon/3$.)

Since it is being assumed that *both*

$$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} f(x) = k \ ,$$

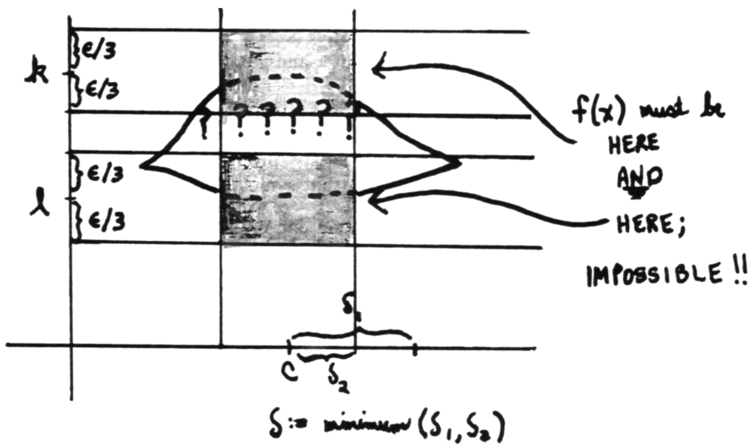
one must be able to get $f(x)$ within $\epsilon/3$ of both l and k, by requiring that x be sufficiently close to c.

So, get a number δ_1 such that whenever x is within δ_1 of c, $f(x)$ must be within $\epsilon/3$ of l.

And, get a number δ_2 so that whenever x is within δ_2 of c, then $f(x)$ must be within $\epsilon/3$ of k.

a contradiction

Take the minimum of δ_1 and δ_2, and call it δ. Then, whenever x is within δ of c, $f(x)$ must be within $\epsilon/3$ of *both* l and k. This is impossible; it is an example of what mathematicians call a *contradiction*. By assuming that k and l are different, one is led to a contradiction. Thus, it must be that k and l are NOT different; that is, they must be equal.



EXERCISE 2

♣ In the preceding argument, the author chose to get the function values $f(x)$ within $\epsilon/3$ of both l and k. Would $\epsilon/2$ have worked? How about $\epsilon/4$? Why do you suppose the author chose $\epsilon/3$?

The following proof ‘formalizes’ the ideas discussed above.

PROOF
limits are unique

Suppose that:

$$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} f(x) = k$$

If $l = k$, we’re done. So suppose that $l \ne k$. Then, there is some positive distance between l and k; call it ϵ. Since ϵ is positive, so is $\epsilon/3$. Since $\lim_{x\to c} f(x) = l$, there exists δ_1 such that whenever $x \in \mathcal{D}(f)$ and $0 < |x - c| < \delta_1$, it must be that $|f(x) - l| < \epsilon/3$.

Since $\lim_{x\to c} f(x) = k$, there exists δ_2 such that whenever $x \in \mathcal{D}(f)$ and $0 < |x - c| < \delta_2$, it must be that $|f(x) - k| < \epsilon/3$.

Take δ to be the minimum of δ_1 and δ_2. Then, for any $x \in \mathcal{D}(f)$ with $0 < |x - c| < \delta$, we must have *both* $|f(x) - l| < \epsilon/3$ and $|f(x) - k| < \epsilon/3$, which is impossible.

Thus, it must be that $k = l$. ∎

EXERCISE 3

♣ Get another calculus book, and look up the *uniqueness of limits* theorem. Compare with what has been discussed here. Is the statement of the theorem the same? Read the proof (slowly and carefully). Is the proof exactly the same? Not every proof uses a contradiction argument. How does the other proof establish that $l = k$?

★★
the logical justification for proof by contradiction

The form of proof, called *proof by contradiction*, is justified by the following logical equivalence:

$$A \Rightarrow B \quad \iff \quad (\text{not}\, B \wedge A) \implies (S \wedge \text{not}\, S) ,$$

where S is any statement.

In the previous proof, the statement A is

$$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} f(x) = k \ ;$$

the statement B is:

$$l = k$$

The contradiction $(S \wedge \text{not}\, S)$ is the fact that $f(x)$ must be IN a certain interval (say, around l) and NOT IN this interval, at the same time.

Next, some rules are developed that tell us many situations in which limits are ‘easy’ to find.

in many cases, evaluating limits is easy;

direct substitution

For many 'nice' functions f, evaluating limits is as easy as *direct substitution*; that is:

$$\lim_{x \to c} f(x) = f(c)$$

This is called *direct substitution* because, to evaluate the limit, one need only substitute the number c into the expression for f.

For example:

$$\lim_{x \to 1} (x^2 - 4) = 1^2 - 4 = -3$$

and

$$\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$$

(Functions that are 'nice' like this are given a special name—they are called *continuous!* This will be studied in more detail in the next section on continuity.)

The next two theorems tell us many 'nice' functions for which evaluating limits is this easy! The numbering scheme (e.g., P1, P2, P3) is merely for easy reference in the exercises and examples.

THEOREM

Properties of Limits

Let b and c denote real numbers; n is a positive integer.

P1) $\lim_{x \to c} b = b$ (The limit of a constant function is the constant.)

P2) $\lim_{x \to c} x = c$

P3) $\lim_{x \to c} x^n = c^n$

some remarks on proving theorems

The proofs of theorems that appear in mathematics books are usually precise, slick, clean, beautiful. Too often, students think that these proofs merely 'jump onto' the paper from the pencils of mathematicians. Not true. Mathematicians rarely 'jump into' a proof. Instead, they *play with* what they're trying to prove. They do things that help them *believe* that it is true. They may 'try out' the theorem in some simple cases, in an attempt to figure out what makes it work.

how you, as a reader, should approach theorems

When you read a theorem, you should do the following:

- Ask yourself: Do I understand what this is telling me that I can DO? Remember, theorems are usually statements of fact. But, *facts can tell you what to do*, if you understand the language.
- Ask yourself: Do I BELIEVE this result? Play with it. Try it in some simple cases. Draw some graphs. *Read and understand the proof.*

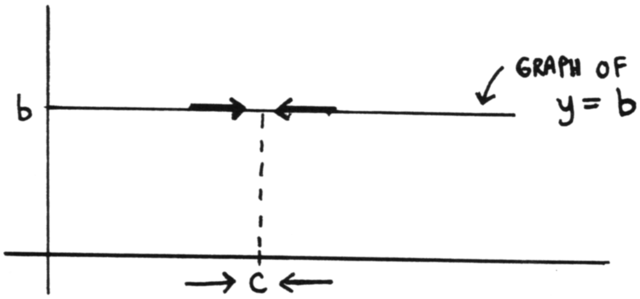
investigating the limit properties of the previous theorem

Let's investigate the properties in the previous theorem, the way a good reader should. Begin with property P1:

$$\lim_{x\to c} b = b$$

What does this say that you can DO? In words, this property states that the limit of a constant function is the constant. It tells you that evaluating the limit of a constant function is easy; just write down the constant.

Next, is this result BELIEVABLE? Recall that the graph of the constant function $f(x) = b$ is a horizontal line, that crosses the y-axis at the number b. No matter *what* the x-value happens to be, the function value is constant at b. Certainly the result is believable.



A precise proof of property P1 must appeal to the definition. It must be shown that one can get the function values as close to b as desired, by requiring that x be sufficiently close to c. Indeed, in this case, no matter what positive number one chooses for ϵ, *any* δ will work. Here's a precise proof:

PROOF of (P1) $\lim_{x\to c} b = b$	Let b and c be real numbers. Choose $\epsilon > 0$, and let $\delta = 1$. If $0 < \|x - c\| < 1$, then $\|b - b\| = 0 < \epsilon$. Thus, $\lim_{x\to c} b = b$. ∎

EXERCISE 4	♣ Are there any other values of δ that would work in the previous proof? Why do you suppose the author chose δ to be 1?

EXERCISE 5

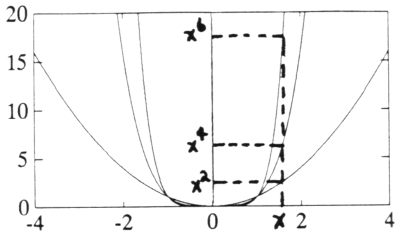
Consider property P2:

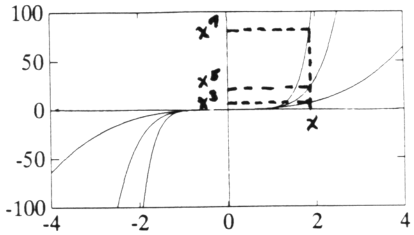
$$\lim_{x\to c} x = c$$

- ♣ 1. What is this telling you that you can DO?
- ♣ 2. Do you believe it? Make a sketch that might help you believe this result.
- ♣ 3. Prove that $\lim_{x\to c} x = c$, by writing down a precise ϵ-δ argument. Use the 4-step process discussed in section 3.2 to find a 'δ that works'.

investigating $\lim_{x\to c} x^n = c^n$

Finding the 'δ that works' is more delicate when investigating $\lim_{x\to c} x^n$, in part due to the fact that different values of c and n will lead to different choices for δ. However, the sketches below certainly make plausible the idea that as x approaches c, x^n must approach c^n.





Next, some *Operations with Limits*.

THEOREM *Operations with Limits*

Let b and c be real numbers; n is a positive integer. Suppose that *both* $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then:

O1) $\lim_{x\to c} bf(x) = b\big[\lim_{x\to c} f(x)\big]$
(You can 'pull constants out' of the limit.)

O2) $\lim_{x\to c} \big[f(x)+g(x)\big] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
(The limit of a sum is the sum of the limits.)

O3) $\lim_{x\to c} f(x)g(x) = \big[\lim_{x\to c} f(x)\big]\big[\lim_{x\to c} g(x)\big]$
(The limit of a product is the product of the limits.)

O4) If $\lim_{x\to c} g(x) \neq 0$, then $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$
(The limit of a quotient is the quotient of the limits.)

O5) $\lim_{x\to c} (f(x))^n = \big[\lim_{x\to c} f(x)\big]^n$
(Power rule)

EXERCISE 6

♣ 1. Does this theorem tell us that

$$\lim_{x\to c}\big[f(x)-g(x)\big] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)\ ,$$

whenever both individual limits exist? Why or why not?

♣ 2. Does this theorem tell us that

$$\lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \lim_{x\to c}\big[f(x)+g(x)\big]\ ,$$

whenever both individual limits exist? Why or why not?

EXERCISE 7

♣ 1. Evaluate the limit:

$$\lim_{x\to 0} x \cdot \frac{1}{x}$$

(Hint: Remember that x is not allowed to equal 0. What is the value of $x \cdot \frac{1}{x}$ for values of x *near* 0?)

♣ 2. Find the flaw in this student's argument.

Student's answer:

By O3:

$$\lim_{x\to 0} x \cdot \frac{1}{x} = \big(\lim_{x\to 0} x\big) \cdot \big(\lim_{x\to 0} \frac{1}{x}\big)$$

Since $\lim_{x\to 0} \frac{1}{x}$ does not exist, it must be that $\lim_{x\to 0} x \cdot \frac{1}{x}$ also does not exist.

investigating the operations with limits

Let's investigate property O1:

$$\lim_{x\to c} bf(x) = b\big[\lim_{x\to c} f(x)\big]$$

the hypotheses of a theorem;

singular: hypothesis

The *hypotheses* of a theorem are the things that are assumed to be true. (Singular: *hypothesis*.) One hypothesis of the previous theorem is that $\lim_{x\to c} f(x)$ exists. Thus, there is some number that $f(x)$ gets close to as x approaches c; in keeping with tradition, let's call this number l. How do the numbers $bf(x)$ differ from $f(x)$? They are each multiplied by b. Thus, as $f(x)$ gets close to l, $bf(x)$ must get close to $b \cdot l$. That is, if

$$\lim_{x\to c} f(x) = l$$

then:

$$\lim_{x\to c} bf(x) = b \cdot l = b \cdot \lim_{x\to c} f(x)$$

So the result does indeed seem plausible.

Similar reasoning should make the remaining operations plausible. We will look at one precise proof, which makes use of the *triangle inequality*, discussed next.

the triangle inequality,

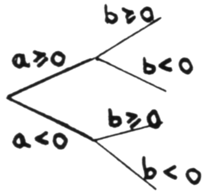
$|a + b| \le |a| + |b|$

Let a and b be real numbers. Then:

$$|a + b| \le |a| + |b|$$

PARTIAL PROOF
of the triangle inequality

Let a and b be real numbers. Since every real number is either nonnegative (≥ 0) or negative (< 0), there are several cases to be considered, as suggested by the 'tree diagram' below.



Recall the precise definition of the absolute value function:

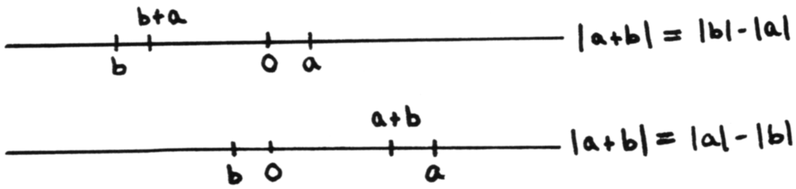
$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Also recall that the number $|x|$ is often called the *magnitude of* x.

Case 1 ($a \geq 0$ and $b \geq 0$). In this case, $|a| = a$ and $|b| = b$. (Why?) Also, since both a and b are nonnegative, so is $a + b$, so that $|a + b| = a + b$. In this case one actually obtains equality:

$$|a + b| = a + b = |a| + |b|$$

Case 2 ($a \geq 0$ and $b < 0$). In this case, writing down all the details often seems to obscure the simple idea, illustrated by the sketches below. The point is that when a and b have different signs, $|a + b|$ is either $|a| - |b|$ (if the magnitude of a is bigger) or $|b| - |a|$ (if the magnitude of b is bigger). But in either case, the difference is less than or equal to $|a| + |b|$.



EXERCISE 8

♣ 1. Write down the proof of

$$|a + b| \leq |a| + |b|$$

in the case when $a < 0$ and $b < 0$. Be sure to write complete mathematical sentences.

♣ 2. Is the case

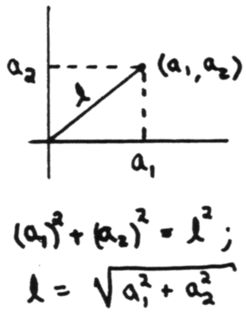
$$a < 0 \text{ and } b \geq 0$$

really any different from the case

$$a \geq 0 \text{ and } b < 0 \text{ ?}$$

Why or why not?

★
Why the name 'triangle inequality'?



★ The triangle inequality also holds when a and b are ordered pairs of real numbers. The 'length' of an ordered pair is found using Pythagorean's theorem:

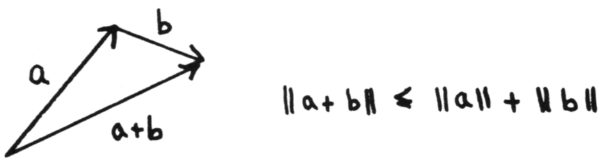
$$\|(a_1, a_2)\| := \sqrt{a_1^2 + a_2^2}$$

Although the absolute value symbol $|\cdot|$ is used to talk about the 'length' (magnitude) of a real number, the *norm* symbol $\|\cdot\|$ is traditionally used to talk about other lengths.

In this setting, the fact that

$$\|a + b\| \le \|a\| + \|b\|$$

has a nice geometric interpretation: in a triangle, the length of a side cannot exceed the sum of the lengths of the remaining two sides. This is the motivation for the name *triangle inequality.*



With the triangle inequality in hand, the precise proof of operation (O2) is now presented.

PROOF of (O2)

that the limit of a sum is the sum of the limits

Suppose that both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, say:

$$\lim_{x\to c} f(x) = l \quad \text{and} \quad \lim_{x\to c} g(x) = k$$

Choose $\epsilon > 0$. Then, $\epsilon/2$ is also positive, and there exists a corresponding δ_1 such that when $x \in \mathcal{D}(f)$ and $0 < |x - c| < \delta_1$, it must be that $|f(x) - l| < \epsilon/2$. (♣ Why?)

Also, there exists δ_2 such that when $x \in \mathcal{D}(g)$ and $0 < |x - c| < \delta_2$, it must be that $|g(x) - k| < \epsilon/2$. (♣ Why?)

Let $\delta := \text{minimum}(\delta_1, \delta_2)$. Then, if $x \in \mathcal{D}(f) \cap \mathcal{D}(g)$ and $0 < |x - c| < \delta$, one obtains:

$$\begin{aligned} |f(x) + g(x) - (l + k)| &= |(f(x) - l) + (g(x) - k)| \\ &\le |f(x) - l| + |g(x) - k| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

This says that:

$$\begin{aligned} \lim_{x\to c} (f(x) + g(x)) &= l + k \\ &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \quad \blacksquare \end{aligned}$$

EXERCISE 9

♣ In the proof above, supply reasons for each of these lines:

$$\begin{aligned} |f(x)+g(x)-(l+k)| &= |(f(x)-l)+(g(x)-k)| && \text{Reason:} \\ &\le |f(x)-l|+|g(x)-k| && \text{Reason:} \\ &< \epsilon/2+\epsilon/2 && \text{Reason:} \\ &= \epsilon && \text{Reason:} \end{aligned}$$

EXERCISE 10

♣ Let a and b be positive numbers. Convince yourself that if $m := \text{minimum}(a,b)$, then $m \le a$ and $m \le b$. (A number line sketch may be all you need to convince yourself of this fact.)

Where was this fact used in the proof of of (O2)?

extending the operations to more than two functions:

Mathematicians realize that facts like

$$\lim_{x\to c}\big[f(x)+g(x)\big] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)\ ,$$

the 'treat it as a singleton' technique

although seemingly holding only for *two* functions, actually hold for *any finite number of functions.* The proof uses a very common 'treat it as a singleton' technique. Assume in what follows that all the individual limits exist.

$$\begin{aligned} \lim_{x\to c} f(x)+g(x)+h(x) &= \lim_{x\to c}\big[f(x)+g(x)\big]+h(x) && \text{(associative law)} \\ &= \lim_{x\to c}\big[f(x)+g(x)\big]+\lim_{x\to c} h(x) && \text{(O2)} \\ &= \lim_{x\to c} f(x)+\lim_{x\to c} g(x)+\lim_{x\to c} h(x) && \text{(O2 again)} \end{aligned}$$

EXERCISE 11

♣ Assuming that all the individual limits exist, show that:

$$\lim_{x\to c} f(x)g(x)h(x) = \big[\lim_{x\to c} f(x)\big]\cdot\big[\lim_{x\to c} g(x)\big]\cdot\big[\lim_{x\to c} h(x)\big]$$

Be sure to write complete mathematical sentences, and give reasons supporting each step in your argument.

In closing, the tools developed in this section are used to show that evaluating limits of ANY polynomial is as easy as direct substitution:

THEOREM

Evaluating limits of polynomials

Let P be any polynomial:

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

Then:

$$\lim_{x\to c} P(x) = P(c)$$

PROOF

$$\begin{aligned}
\lim_{x\to c} P(x) &= \lim_{x\to c}\,(a_n x^n + \cdots + a_1 x + a_0) && \text{(definition of } P\text{)}\\
&= \lim_{x\to c} a_n x^n + \cdots + \lim_{x\to c} a_1 x + \lim_{x\to c} a_0 && \text{(O2)}\\
&= a_n \lim_{x\to c} x^n + \cdots + a_1 \lim_{x\to c} x + a_0 && \text{(O1) and (P1)}\\
&= a_n c^n + \cdots + a_1 c + a_0 && \text{(P3)}\\
&= P(c) && \text{(polynomial } P\text{, evaluated at } c\text{)}
\end{aligned}$$

EXAMPLE

For example:

$$\lim_{x\to 1}(x^2 - 3x + \sqrt{2}) = 1^2 - 3(1) + \sqrt{2} = -2 + \sqrt{2}$$

QUICK QUIZ
sample questions

1. Explain, in a couple English sentences, how a mathematician often shows that an object is UNIQUE.
2. Under what condition(s) is the limit of a sum equal to the sum of the limits?
3. Give a precise statement of the 'triangle inequality' for real numbers.
4. Suppose you are told that, for a given function f and constant c, 'evaluating the limit $\lim_{x\to c} f(x)$ is as easy as direct substitution'. What does this mean?
5. Suppose that:

$$\lim_{x\to 1} f(x) = 3,\quad \lim_{t\to 1} g(t) = 5,\quad \text{and}\quad \lim_{y\to 1} h(y) = 2$$

Can you evaluate the following limit?

$$\lim_{z\to 1} \frac{-2f(z) + g(z)}{h(z)}$$

If so, do it.

KEYWORDS
for this section

Existence and uniqueness arguments, the end-of-proof symbol ▮, *uniqueness of limits, direct substitution, properties of limits, how you should approach theorems, operations with limits, the triangle inequality, extending operations to more than two functions, limits of polynomials.*

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN).

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

1. If $\lim_{x\to c} f(x) = l$ and $\lim_{y\to c} f(y) = m$, then $l = m$.
2. If $\lim_{t\to c} f(t) = q$ and $\lim_{x\to c} f(x) = r$, then $q = r$.
3. $\lim_{x\to c} f(x) = \lim_{y\to c} f(y)$
4. $\lim_{x\to c} f(x) = \lim_{x\to d} f(x)$
5. If $\epsilon > 0$, then $\frac{\epsilon}{2} > 0$.
6. If $\frac{\epsilon}{2} > 0$, then $\epsilon > 0$.
7. $\epsilon > 0 \quad \iff \quad \frac{\epsilon}{2} > 0$
8. $\epsilon > 0 \quad \iff \quad 2\epsilon > 0$
9. $\epsilon > 0 \quad \iff \quad (\epsilon - .1) > 0$
10. $\lim_{x\to c} d = d$ (Here, it is assumed that c and d are real numbers.)
11. $\lim_{x\to 2} x^{100} = 2^{100}$
12. $\lim_{y\to -1} y = -1$
13. $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
14. If the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.

For the remaining problems, suppose that:

$$\lim_{x\to c} f(x) = -1, \quad \lim_{x\to c} g(x) = 2, \quad \text{and} \quad \lim_{x\to c} h(x) = 0$$

If possible, evaluate the following limits. If you don't have enough information to evaluate the limit, so state. Be sure to write complete mathematical sentences.

15. $\lim_{t\to c}[f(t) + g(t)]$
16. $\lim_{t\to c}(f - g)(t)$
17. $\lim_{y\to d}[f(y)g(y)]$
18. $\lim_{x\to c}\big([3g(x) - f(x)] \cdot h(x)\big)$