9. MATHEMATICIANS ARE FOND OF COLLECTIONS

collections

'Collections' are extremely important in life: when we group together objects that are in some way similar, then it is easier to talk about the unit. The list of possible 'collections' goes on and on: females, democrats, students at Miss Hall's School, your relatives, the folders in your personal filing cabinet, your favorite books, ...

Collections are also extremely important in mathematics. A group of similar objects can be given a name, making the group easier to refer to. Tools can be developed for working with the objects in a particular collection. Furthermore, you've probably noticed the fondness that mathematicians have for using letters (like x): every time you see such a letter, there's a 'collection' associated with the letter lurking in the background. (More on this in the next section—**Holding This, Holding That**.) In mathematics, people study collections of *numbers* (like $\mathbb{R}$ and $\mathbb{Z}$); collections of *sentences*; even collections of collections! The idea of 'collection' is made precise by the mathematical construct called a *set*.

DEFINITION
set

A *set* is a collection with the following property: given *any* object, either the object *is* in the collection, or *isn't* in the collection.

understanding the definition

The key idea is this: to qualify as a set, one need only be certain that *every* object (like the number 2, or 'chair', or 'grasshopper') is either IN the collection, or NOT IN the collection. It's not necessary to know *which* of these two cases occurs (i.e., whether the object is IN or NOT IN the collection); it's only necessary to know that *exactly one* of these two situations occurs!

This idea is a bit subtle; a couple of examples should provide some clarification.

EXAMPLE
a non-set

'*The collection of some people*' is not a set. Is the author of this book in the collection? Maybe. Or maybe not. That is, given the object 'author of this book' (or any other person, for that matter), it is impossible to state with certainty that either the author IS in the collection, or IS NOT in the collection. Roughly, 'vagueness' prevents this collection from being a set.

EXAMPLE
a set

Consider the collection of numbers having $3, 6, 9, 12, 15, 18, \ldots$ as members. Observe that 3 goes into each of these numbers evenly; and, there are infinitely many members in this collection. Is this a set? That is, given any object, can we definitively say that either it IS in the collection, or IS NOT in the collection? Let's try a few:

Is 'grasshopper' in the collection? Certainly not: it's not even a number, so it doesn't have a chance.

Is 7 in the collection? Well, it's a number, but 3 doesn't go into it evenly. So, it's not in the collection.

Is 72 in the collection? It's a number, and 3 goes into it evenly, so it is in the collection.

Of course, it's impossible to 'test' all possible objects. However, we CAN with certainty conclude that, given *any* object, either it's IN the collection, or it ISN'T. For example, consider the (very large) number

$$35,983,205,119,780,238,482,108,222,239,407,290,981,239\ . \qquad (*)$$

Is this number in the collection? You can't use your calculator to help you decide, because the number is way too large. And (unless you're a *very* patient person) you probably don't want to take the time to divide it by 3, by hand. (You could, of course, use the divisibility test discussed in section 2, but even applying this test requires some patience.) However, we CAN with complete certainty say the following: either 3 goes in evenly, or it doesn't. One or the other must happen. So, the collection is a set.

mathematics is primarily a written language

One part of the previous example brings to light a subtle fact about mathematics: it has primarily evolved to be a *written* language, not a *spoken* language. Consequently, people sometimes run across mathematical stuff that is not convenient to read aloud. The number in (*) is one such case. Although standard vocabulary allows us to read many large numbers aloud, (*) is so large that it goes beyond the words supplied. If forced to read (*) aloud, then most people would either change its name to one more suited to large numbers (scientific notation), or else say: 'the large number whose digits are: three, five, comma; nine, eight, three, comma; ... ' .

a set is an EXPRESSION, not a sentence

Whenever you are presented with any new mathematical concept, you should immediately address its most primary classification: is it an *expression*, or a *sentence*? **A set is a mathematical expression**: it is a name given to some collection of interest. It doesn't make sense to ask if a set is TRUE or FALSE, because a set is not a sentence. Sets (like expressions in general) can have lots of different names. The remainder of this section is devoted to notation used in connection with sets.

members of a set; elements of a set

finite set; infinite set

The objects in a set are called its **elements**, or its **members** (the two terms are used interchangeably). A set can have no members (0 members), 1 member, 2 members, 3 members, etc. If a set has n members, where n is a whole number, then it is called a **finite** (FI–nite) set. Otherwise, it is called an **infinite** (IN–fi–nit) set.

For example, a set with 203 members is a finite set. A set with no members is a finite set. The set of integers that lie between -3 and 985 is a finite set. The set of *all* integers is an infinite set. The set of real numbers is an infinite set. The set of all real numbers between 2 and 3 is an infinite set.

EXERCISES

1. Sometimes a definition is embedded in a paragraph, instead of putting it in a nice box labeled **DEFINITION**. This just happened. You were actually given four definitions: element (of a set), member (of a set), finite set, infinite set. State the definitions of these four things.

2. Consider the sentence:

   'n is a whole number.'

   This sentence can be true or false, depending upon the number chosen for n. What value(s) of n make the sentence true? False?

list method for naming (some) sets

Some sets (but not all) can be easily described using the *list* method: in this method, the members are separated by commas, and enclosed in braces $\{\ \ \}$.

the order of the members in the list doesn't make a difference

For example, the set $\{0,1,2\}$ has three members: 0 is a member, 1 is a member, and 2 is a member. When using the list method with a finite number of elements, the order in which the elements are listed doesn't make any difference. Therefore,

$$\{0,1,2\} \text{ and } \{0,2,1\} \text{ and } \{1,0,2\} \text{ and } \{1,2,0\} \text{ and } \{2,0,1\} \text{ and } \{2,1,0\}$$

are all just different names for the same set. Notice that there are $3\cdot 2\cdot 1 = 6$ ways to rearrange the three elements in this set.

How is '$\{0,1,2\}$' read aloud?

You can read '$\{0,1,2\}$' as:

- 'the set with members 0, 1, and 2'; or
- 'open brace, zero, one, two, close brace'.

The first way expresses an understanding of the symbols; the second is a 'literal' reading of each symbol.

EXERCISES

3. It's important to be able to write braces { } correctly. (Braces are 'curly', like the things on teeth.) In particular, braces must be easy to distinguish from parentheses ( ) and brackets [ ]. Trace the following as practice:

{ } { } { } { } { } { }

( ) ( ) ( ) ( ) ( ) ( )

[ ] [ ] [ ] [ ] [ ] [ ]

4. What do the centered dots in the expression $3\cdot 2\cdot 1$ mean?

5. How many members are in the set $\{a,b,c,d,e\}$? Make a conjecture (i.e., an educated guess) as to how many different re-arrangements of this set there are.

A set can have lots of different names!

The set $\{\dots,\ -3,-2,-1,0,1,2,3,\ \dots\}$ has an infinite number of members. Indeed, this is the set of integers, denoted by the symbol $\mathbb{Z}$. Since $\mathbb{Z}$ and $\{\dots,\ 3,-2,-1,0,1,2,3,\ \dots\}$ are just different names for the same set, the sentence

$$\mathbb{Z} = \{\dots,\ -3,-2,-1,0,1,2,3,\ \dots\}$$

is true.

there are sets that can't be described using the list method

Not all sets can be described using the list method. For example, the real numbers shaded below can't be described using the list method. You could certainly try: $\{2\,,\ 2.001\,,\ 2.002\,,\ 2.003\,,\ \dots\,,\ 3\}$; but you'd be missing infinitely many numbers between, say, 2.001 and 2.002.

a verb to discuss membership in a set: $\in$

To talk about membership in a set, we need the verb $\in$, which is called the '*is in*' or '*is an element of*' or '*is a member of*' symbol. A precise discussion of this verb follows.

sentence:
$x \in S$

Let x represent *any* object, and let S represent *any* set. The sentence

$$x \in S$$

is read as:

- '$\,$ex is in S' or
- '$\,$ex is an element of S' or
- '$\,$ex is a member of S'.

These three phrases are used interchangeably.

naming conventions for sets:

sets are named with capital letters;

uppercase and lowercase are NOT interchangeable

The sentence '$x \in S$' illustrates a couple naming conventions for sets. Firstly, the letter S is commonly used to name sets, since it is the first letter in the word 'set'. Secondly, sets are usually represented by uppercase (capital) letters—more on this in the section **Holding This, Holding That**. Furthermore, this is a good time to mention that, in mathematics, uppercase and lowercase letters are NOT interchangeable: the lowercase (like t) and uppercase (like T) versions of letters usually represent totally different objects.

When is the sentence '$x \in S$' true? False?

When is the sentence '$x \in S$' true? False?

If x really IS a member of S, then the sentence '$x \in S$' is TRUE. And, if the sentence '$x \in S$' is TRUE, then x must be a member of S.

Similarly, the sentence '$x \in S$' is FALSE precisely when x IS NOT a member of S.

Before looking at some examples, it's necessary to point out a sentence structure that is commonly used in mathematics when something is being given a name.

What does the phrase 'LET $S = \{1,2,3\}$' mean?

In mathematics, the phrase

$$\text{LET } S = \{1,2,3\}$$

means: take the set $\{1,2,3\}$ and give it the name S, so that it will be easier to refer to. More generally, a sentence of the form

$$\text{LET } \mathit{NAME} = \mathit{EXPRESSION}$$

is used to give the name *NAME* to the expression *EXPRESSION*. **The word 'LET' is the key to knowing that something is being named.** Here are some examples:

- Let $x = 4.217$. (The name x is being given to the number 4.217.)
- Let $W = \{0,1,2,3,\dots\}$. (The name W is being given to the set of whole numbers.)
- Let $t = \frac{1}{2} + \frac{1}{3}$. (The name t is being given to the sum of $\frac{1}{2}$ and $\frac{1}{3}$.)

EXAMPLE
using the verb '$\in$'

Here are some examples using the verb '$\in$'. Let $S = \{1,2,3\}$. Then, the following sentences are all true:

$$1 \in S \qquad 2 \in S \qquad 3 \in S \qquad \tfrac{6}{3} \in S \qquad \tfrac{16-7}{3} \in S$$

(Note: $\frac{6}{3}$ is just another name for 2; and $\frac{16-7}{3}$ is just another name for 3.)

The following sentences are false:

$$4 \in S \qquad 1.000001 \in S \qquad 0 \in S$$

EXERCISES

6. In mathematics, how would you say: 'Take the set $\{a,b,c,d,e\}$, and give it the name T'?

7. In mathematics, how might you more compactly say: 'Take all the whole numbers greater than or equal to 7, and give this collection the name S'?

EXERCISES

8. Let $W = \{3, 4, 5, \dots\}$. Decide whether the following sentences are true, false, or sometimes true/sometimes false (ST/SF):

(a) $3 \in W$

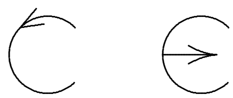
(b) $107 \in W$

(c) $\frac{8}{3} \in W$

(d) $\frac{9}{3} \in W$

(e) $x \in W$

Step 1: *Step 2:*



9. It's important to be able to write the symbol $\in$ correctly. Trace the following as practice:

$\in \quad \in \quad \in \quad \in \quad \in \quad \in \quad \in \quad \in \quad \in \quad \in \quad \in$

10. Classify each entry below as an expression or a sentence.

If an expression, state whether it's a number or a set.

If a sentence, state how you might read it aloud, and state whether it is true, false, or ST/SF.

(a) 5

(b) $\{5\}$

(c) $\mathbb{Z}$

(d) $5 \in \mathbb{R}$

(e) $5.1 \in \mathbb{Z}$

the 'not in' verb: $\notin$

the sentence: $x \notin S$

A verb is frequently negated by putting a slash through it. Consequently, the sentence '$x \notin S$' is read as 'ex is not in S' or 'ex is not an element of S' or 'ex is not a member of S'.

The sentence '$x \notin S$' is true when x is not a member of S; it is false otherwise.

For example, the sentence '$5 \notin \{1, 2, 3\}$' is true.

a slash / is used to negate action

Here are more examples of using a slash to negate a verb:

verb	how to read	negated verb	how to read
$\in$	is in	$\notin$	is not in
$\in$	is an element of	$\notin$	is not an element of
$\in$	is a member of	$\notin$	is not a member of
$=$	is equal to	$\neq$	is not equal to
$=$	equals	$\neq$	does not equal
$>$	is greater than	$\not>$	is not greater than
$<$	is less than	$\not<$	is not less than

Notice the variety of ways that some symbols can be read.

EXERCISE

11. For each sentence below, make a number line, and shade the value(s) of x that make the sentence true. Be careful to distinguish between hollow dots (numbers not included) and solid dots (numbers included).

(a) $x \in \{0, 3, 5\}$

(b) x is a real number and $x \neq 2$

(c) x is a real number and $x \notin \{0, 3, 5\}$

the empty set has no members:
$\emptyset$
$\{\ \}$

There is exactly one set that has NO members: it is appropriately called the *empty set*, and is denoted using either the symbol $\emptyset$, or a pair of braces with nothing inside: $\{\,\}$. Consequently, the sentence $x \in \emptyset$ (or $x \in \{\,\}$) is always FALSE, since the empty set has no members!

The astute reader may have noticed the similarity between the symbols 0̸ (sometimes used for the number zero) and $\emptyset$ (the empty set). Context will help to clarify the correct interpretation, since numbers and sets get used in different types of places.

EXERCISE

12. State how you might read each sentence. Also, classify each sentence as true, false, or ST/SF:

(a) $1 \in \{\ \}$

(b) $0 \in \emptyset$

(c) $0 \notin \{\ \}$

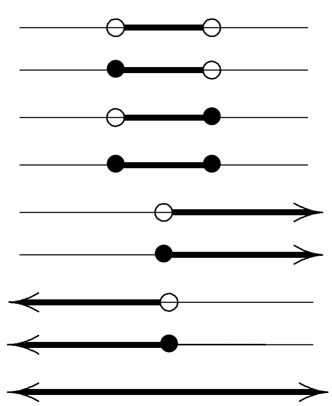
(d) $x \notin \emptyset$

intervals

As mentioned earlier, not all sets can be listed. Indeed, there is an important class of frequently-used sets, called *intervals*, that cannot be listed. The definition of an *interval* is given next, and then an important notation used to describe intervals.

DEFINITION
interval

An *interval* is a set of real numbers that has one of the following forms:



two endpoints, neither endpoint included

two endpoints, only left-hand endpoint included

two endpoints, only right-hand endpoint included

two endpoints, both endpoints included

one endpoint, not included, with everything to its right

one endpoint, included, with everything to its right

one endpoint, not included, with everything to its left

one endpoint, included, with everything to its left

Everything!

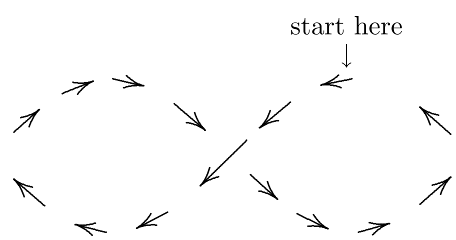
★

The empty set and singletons are sometimes considered to be intervals. The empty set is a 'degenerate' form of an open interval (a,b), when $a = b$. A singleton is a 'degenerate' form of a closed interval $[a,b]$, when $a = b$.

interval notation:

∞ *'infinity'*
$-\infty$ *'negative infinity'*

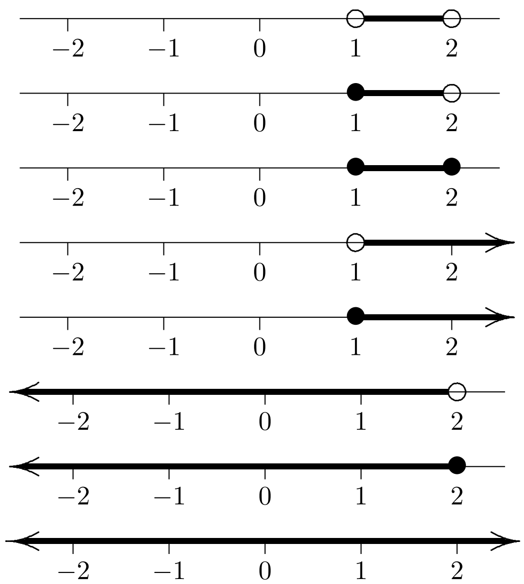
Intervals are very common in mathematics, so there is a special notation for naming them, which is appropriately called *interval notation*. Notice carefully the difference between the use of parentheses ( ) and brackets [ ] in the following examples. The symbol ∞ is read as 'infinity' (in–FIN–i–tee), and the symbol $-\infty$ is read as 'negative infinity'.



Drawing the 'infinity' symbol

examples of interval notation

SET NAME USING INTERVAL NOTATION:



$(1,2)$

$[1,2)$

$[1,2]$

$(1,\infty)$

$[1,\infty)$

$(-\infty,2)$

$(-\infty,2]$

$(-\infty,\infty)$

rules for interval notation

Here are the rules for interval notation:

WHEN THERE ARE TWO ENDPOINTS:

- List the endpoints of the interval, separated by commas; always list the endpoints in order from left to right on the number line.
- If an endpoint is to be INCLUDED, put a bracket [ ] next to it.
- If an endpoint is NOT to be included, put a parenthesis ( ) next to it.

WHEN THERE IS ONLY ONE ENDPOINT:

- The symbol ∞ is used when you are to continue forever to the right.
- The symbol $-\infty$ is used when you are to continue forever to the left.
- Note that ∞ is NOT a real number: that is, there is no point on the number line corresponding to ∞. Instead, ∞ suggests the *idea* that the number line extends infinitely far to the right: given any real number (no matter how far to the right of zero it lives), there is always a real number that lives farther to the right! (What does $-\infty$ suggest?) Consequently, parentheses ( ) are always used with the symbols ∞ or $-\infty$, since they can't be included.

∞ is NOT a real number!

MEMORY DEVICE

Brackets [ ] have sharp corners—and dust collects in corners—so brackets correspond to FILLED-IN endpoints. (Imagine the endpoint filled with dust!) On the other hand, parentheses ( ) do NOT have corners, so dust can't collect here; parentheses correspond to HOLLOW endpoints.

reading interval notation

Unfortunately, there's not always a great way to read interval notation aloud—another illustration that mathematics is primarily a written language, not a spoken language. Here are some possibilities at this stage in your mathematical career:

line #	INTERVAL	POSSIBLE WAY TO READ ALOUD
1	$(2,3)$	the real numbers between 2 and 3, not including the endpoints
2	$(2,3)$	parenthesis . . . 2 . . . comma . . . 3 . . . parenthesis
3	$[2,3)$	the real numbers between 2 and 3, including 2 but not 3
4	$[2,3)$	bracket . . . 2 . . . comma . . . 3 . . . parenthesis
5	$(2,\infty)$	the real numbers greater than 2
6	$[2,\infty)$	the real numbers greater than or equal to 2
7	$(-\infty,2)$	the real numbers less than 2
8	$(-\infty,2]$	the real numbers less than or equal to 2

Lines 1 and 3 display an understanding of the symbols; lines 2 and 4 are 'literal' or 'verbatim' readings of the symbols. Lines 5 through 8 use the words 'greater than' and 'less than', which are thoroughly discussed in the section **I Live Two Blocks West Of You**.

EXERCISE

13. Classify each entry below as an expression or a sentence. If an expression, state whether it is a number or a set. If a sentence, state how you might read it aloud, and state whether it is true, false, or ST/SF.

(a) $\{1,2\}$
(b) $[1,2]$
(c) $1+2$
(d) $(1,2]$
(e) $1 \in (1,2]$
(f) $1 \in [1,2)$

set-builder notation

There are sets that cannot easily be described using either the list method, or interval notation. In such cases, a naming scheme called 'set-builder notation' usually comes to the rescue. (Set-builder notation will not be discussed in this book.)

subset

Sometimes, it is necessary to discuss various *subcollections* chosen from a given set. This idea of *subcollection* is made precise as follows:

DEFINITION
subset

Let S be a set. Set B is called a *subset* of S if any one of the following three conditions holds:

(a) B is the set S itself
(b) B is the empty set
(c) each member of B is also a member of S

understanding the definition

The first sentence,

'Let S be a set.'

means

'Let S be *any* set.'

So why don't mathematicians say 'any', if they mean 'any'? Well, the word 'a' is shorter than the word 'any'—and mathematicians are an extremely frugal lot.

investigating (a)–(c)

Now, let's investigate conditions (a)—(c). Remember that these are the conditions under which B gets to be called a subset of S.

Condition (a) tells us that a set is a subset of itself; this is the 'subcollection' consisting of *everything* in the original set.

Condition (b) tells us that the empty set is a subset of every set; this is the 'subcollection' consisting of *nothing* from the original set.

By combining conditions (a) and (b), we see that every set (except the empty set) is guaranteed to have at least two subsets: itself and the empty set. The next example explores condition (c), and shows that most sets have *lots* of subsets:

EXAMPLE

Let $S = \{1, 2, 3\}$. All the subsets of S are listed below:

		$\{1, 2, 3\}$	(S itself)
		$\{\ \}$	(the empty set)
$\{1\}$	$\{2\}$	$\{3\}$	(all one-member subsets)
$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	(all two-member subsets)

Thus, the set $\{1, 2, 3\}$ (or *any* set with three members) has eight subsets.

EXERCISES

14. List all the subsets of $\{a, b\}$. How many subsets are there?
15. List all the subsets of $\{0, 2, 4\}$. How many subsets are there?
16. Justify your answers to each of the following questions:
    (a) Is $\{-1, 2, 3\}$ a subset of $\mathbb{R}$?
    (b) Is $\{-1, 2, 3\}$ a subset of the whole numbers?
    (c) Is $\{-1, 2, 3\}$ a subset of the integers?
    (d) Is $\{-1, 2, 3\}$ a subset of $(-2, \infty)$?

END-OF-SECTION EXERCISES

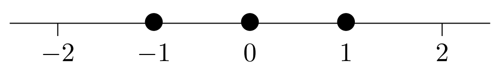
For exercises 17–23: Classify each entry as a mathematical expression (EXP), or a mathematical sentence (SEN).

If an expression, state whether it is a number or a set.

Classify the truth value of each sentence: (always) true (T); (always) false (F); or sometimes true/sometimes false (ST/SF).

17. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ ______________
18. $\frac{1}{100} \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ ______________
19. $0.01 \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ ______________
20. $(3, 5]$ ______________
21. $3 \in (3, 5]$ ______________
22. $5 \in (3, 5]$ ______________
23. $4.997 \in (3, 5]$ ______________

Describe the following sets of numbers using correct set notation. Use either list or interval notation; whichever is appropriate.

24.


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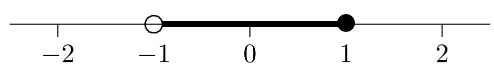

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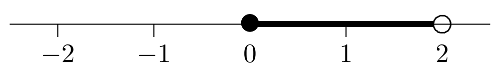

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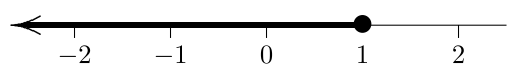

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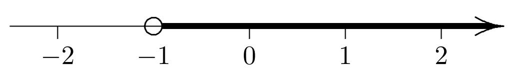


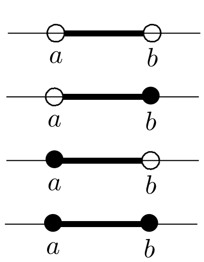
29. List all subsets of the set in exercise (24).

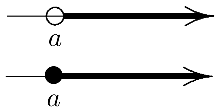
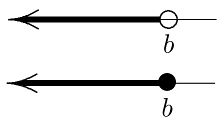
30. Is the set of positive integers a subset of the set in exercise (28)? Justify your answer.

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SECTION SUMMARY
MATHEMATICIANS ARE FOND OF COLLECTIONS

NEW IN THIS SECTION	HOW TO READ	MEANING
set		A collection satisfying: given *any* object, either the object *is* is the collection, or *isn't* in the collection. A set is an *expression*.
members; elements		the objects in a set
finite set	FI–nite	a set with n members, where n is a whole number
infinite set	IN–fi–nit	a set that is not finite
list method		a method for naming sets whose elements can be listed
{ , }	open brace; close brace; plural is 'braces'	used in list notation; the elements are listed, separated by commas, inside the braces
$\in$	'is in' 'is an element of' 'is a member of'	verb used to talk about membership in a set
$x \in S$	'x is in S' 'x is an element of S' 'x is a member of S'	sentence: true when x is a member of the set S; false otherwise
Let *NAME* = *EXPRESSION*		used whenever you want to assign the name *NAME* to the expression *EXPRESSION*
/	(forward) slash	used to negate a verb
$x \notin S$	'x is not in S'	sentence: true when x is not a member of S; false otherwise
$\emptyset$ or { }	the empty set	the unique set that has no members
(a,b) $(a,b]$ $[a,b)$ $[a,b]$ 	the real numbers between a and b (with various endpoints included/not included)	intervals of real numbers

NEW IN THIS SECTION	HOW TO READ	MEANING
(a, ∞) $[a, \infty)$ 	the real numbers greater than a; greater than or equal to a	intervals of real numbers
$(-\infty, b)$ $(-\infty, b]$ 	the real numbers less than b; less than or equal to b	intervals of real numbers
(,)	open parenthesis; close parenthesis; plural is 'parentheses'	when used in interval notation, denotes that an endpoint is NOT to be included; a parenthesis is always used next to ∞ or $-\infty$
[,]	open bracket; close bracket; plural is 'brackets'	when used in interval notation, denotes that an endpoint IS to be included
∞ , $-\infty$	infinity; negative infinity	The symbol ∞ suggests the idea that given any real number, no matter how far to the right of zero, there is always one farther to the right. The symbol $-\infty$ suggests the idea that given any real number, no matter how far to the left of zero, there is always one farther to the left.
subset		Set B is a subset of a set S if one of the following conditions holds: $B = S$, $B = \emptyset$, or each member of B is also a member of S.