33. SOLVING LINEAR INEQUALITIES IN ONE VARIABLE

linear inequalities in one variable

The only difference between a linear *equation* in one variable and a linear *inequality* in one variable is the verb: instead of an '$=$' sign, there is an inequality symbol ($<$, $>$, $\le$, or $\ge$). One new idea is needed to solve linear inequalities: if you multiply or divide by a negative number, then the direction of the inequality symbol must be changed. This idea is explored in the current section. We begin with a definition:

DEFINITION
linear inequality in one variable

A *linear inequality in one variable* is a sentence of the form

$$ax + b < 0\,, \quad a \ne 0\,.$$

The inequality symbol can be any of the following: $<$, $>$, $\le$, or $\ge$.

EXERCISES

1. What does the restriction $a \ne 0$ tell you in this definition?
2. Decide if each sentence is a linear inequality in one variable. If not, give a reason why.
   (a) $2x - 5 \le 0$
   (b) $1 + 2x > 6x + \frac{1}{2}$
   (c) $x^2 - x \ge 3$
   (d) $0.4t - 7 < 2t$
   (e) $\dfrac{1}{t} + t \ge 0$
   (f) $5.4(x - \frac{1}{3}) \le x - 0.2(1 + 8x)$
   (g) $3x < 2y + 1$

Here is a precise statement of the tools for solving inequalities. Try translating them yourself, before reading the discussion that follows. Think: "What do these *facts* tell me that I can *do*?"

THEOREM
tools for solving inequalities

For all real numbers a, b, and c,

$$a < b \iff a + c < b + c\,.$$

If $c > 0$, then

$$a < b \iff ac < bc\,.$$

If $c < 0$, then

$$a < b \iff ac > bc\,.$$

The inequality symbol may be $<$, $>$, $\le$, or $\ge$, with appropriate changes made to the equivalence statements.

translating the theorem:

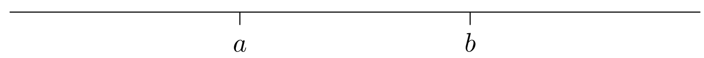
you can add (or subtract) the same number to (or from) both sides of an inequality, and this won't change its truth

The first sentence, '$a < b \iff a + c < b + c$', holds for all real numbers a, b, and c. This says that you can add (or subtract) the same number to (or from) both sides of an inequality, and it won't change the truth of the inequality.

Here's the idea: if a lies to the left of b on a number line, and both numbers are translated by the same amount c, then $a + c$ still lies to the left of $b + c$.

EXERCISES

3. Numbers a and b are shown on the number line below. (You may assume that 1 unit is about $\frac{1}{4}$ inch.)



(a) Write an inequality that is true for a and b.

(b) Clearly label the numbers $a + 1$ and $b + 1$ on the number line. Write an inequality that is true for $a + 1$ and $b + 1$.

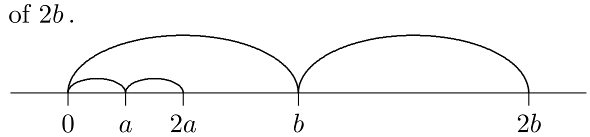
(c) Clearly label the numbers $a - 1$ and $b - 1$ on the number line. Write an inequality that is true for $a - 1$ and $b - 1$.

translating the theorem:

you can multiply (or divide) both sides of an inequality by the same positive number, and this won't change its truth

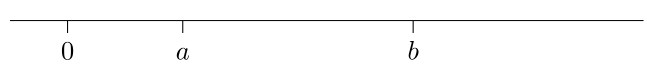
The second sentence, '$a < b \iff ac < bc$', only holds for $c > 0$. This says that you can multiply (or divide) both sides of an inequality by the same *positive* number, and it won't change the truth of the inequality.

Here's the idea. Think about the situation when $c = 2$. If a lies to the left of b on a number line, and we double both number's distance from 0, then $2a$ still lies to the left of $2b$.



EXERCISES

4. Numbers a and b are shown on the number line below.



(a) Write an inequality that is true for a and b.

(b) Clearly label the numbers $\frac{1}{2}a$ and $\frac{1}{2}b$ on the number line. Write an inequality that is true for $\frac{1}{2}a$ and $\frac{1}{2}b$.

(c) Clearly label the numbers $1.5a$ and $1.5b$ on the number line. Write an inequality that is true for $1.5a$ and $1.5b$.

translating the theorem:

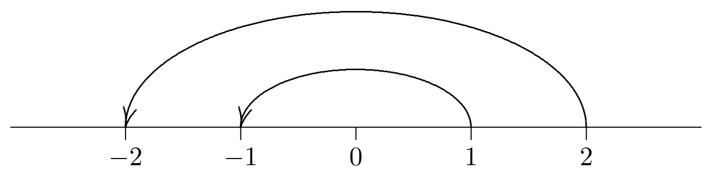
if you multiply (or divide) both sides of an inequality by the same negative number, you must change the direction of the inequality symbol

It's the third sentence where something interesting is happening.

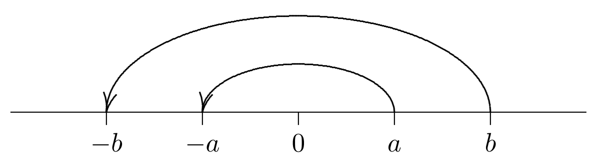
The third sentence, '$a < b \iff ac > bc$', holds for $c < 0$. Notice the '$<$' symbol in the sentence '$a < b$', but the '$>$' symbol in the sentence '$ac > bc$'. This says that if you multiply (or divide) both sides of an inequality by the same *negative* number, then the direction of the inequality symbol must be changed in order to preserve the truth of the inequality.

Let's look at two examples, to begin to understand this situation.

The sentence '$1 < 2$' is true. Multiplying both sides by -1 and changing the direction of the inequality gives the new sentence '$-1 > -2$', which is still true. The number 1 lies to the left of 2, and the opposite of 1 lies to the right of the opposite of 2. The process of 'taking the opposite' of two numbers changes their positions relative to each other on the number line.



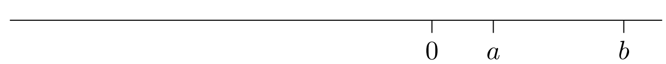
Here's a second example. In the sketch below, '$a < b$' is true, because a lies to the left of b. Multiplying both sides by -1 sends a to its opposite $(-a)$ and sends b to its opposite $(-b)$. Now, the opposite of a is to the right of the opposite of b. That is, $-a > -b$.



This simple idea is the reason why you must flip the inequality symbol when multiplying or dividing by a negative number.

EXERCISES

5. Numbers a and b are shown on the number line below.



(a) Write an inequality that is true for a and b.

(b) Clearly label the numbers $-2a$ and $-2b$ on the number line. Write an inequality that is true for $-2a$ and $-2b$.

(c) Clearly label the numbers $-\frac{1}{2}a$ and $-\frac{1}{2}b$ on the number line. Write an inequality that is true for $-\frac{1}{2}a$ and $-\frac{1}{2}b$.

the theorem holds for other inequality symbols

Although the theorem is stated using the inequality symbol ' $<$ ', it also holds for all other inequality symbols.

Here's the statement using the ' $\geq$ ' symbol:

For all real numbers a, b and c,

$$a \geq b \iff a + c \geq b + c\,.$$

If $c > 0$, then

$$a \geq b \iff ac \geq bc\,.$$

If $c < 0$, then

$$a \geq b \iff ac \leq bc\,.$$

EXERCISES

6. Give the statement of the theorem using the ' $\leq$ ' symbol.
7. Let a, b and c be real numbers. What does the fact
$$a < b \iff a + c < b + c$$
tell you that you can DO? Do not use any variable (like c) when giving your answer.
8. Let a, b and c be real numbers, with $c < 0$. What does the fact
$$a > b \iff ac < bc$$
tell you that you can DO? Do not use any variable (like c) when giving your answer.

steps for solving a linear inequality

The basic steps for solving a linear inequality in one variable are outlined next. They are identical to the thought process for solving linear equations, with the new idea of changing the direction of the inequality if you multiply or divide by a negative number.

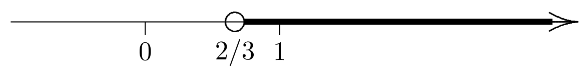
- Simplify both sides of the inequality as much as possible. In particular, combine like terms.
- Get all the x terms on one side (usually the left side).
- Get all the constant terms on the other side (usually the right side).
- Get x all by itself, by multiplying or dividing by an appropriate number. Change the direction of the inequality if you multiply or divide by a negative number.

EXAMPLE
solving a linear inequality

Solve: $-3x + 1 < 4x - 5 + 2x$. Shade the solution set on a number line.

$$-3x + 1 < 4x - 5 + 2x \quad \text{(original inequality)}$$
$$-3x + 1 < 6x - 5 \quad \text{(simplify)}$$
$$-9x + 1 < -5 \quad \text{(subtract } 6x \text{ from both sides)}$$
$$-9x < -6 \quad \text{(subtract 1 from both sides)}$$
$$x > \frac{-6}{-9} \quad \text{(divide both sides by } -9\text{; change direction of inequality)}$$
$$x > \frac{2}{3} \quad \text{(simplify)}$$

The solution set is shaded below:



the solution set of an inequality versus the solution set of an equation

Notice that there are an infinite number of solutions to a linear inequality in one variable, which is very different from the unique solution to a linear equation in one variable. Therefore, you certainly can't check every solution to a linear inequality!

spot-checking

However, there is a way to gain confidence in your answer and catch mistakes, called 'spot-checking'. Here's the general procedure, followed by a 'spot-check' of the previous example:

- Choose a simple number that makes the final inequality true. Substitute it into the original inequality, which should also be true.
- Choose a simple number that makes the final inequality false. Substitute it into the original inequality, which should also be false.

a spot-check of the previous example

In the previous example, the inequality

$$-3x + 1 < 4x - 5 + 2x \quad \text{(the 'original' inequality)}$$

was transformed to

$$x > \tfrac{2}{3} \quad \text{(the 'final' inequality).}$$

Here's a 'spot-check' for this example:

- Choose a simple number that makes the final inequality '$x > \frac{2}{3}$' true: choose, say, $x = 1$. Substitution into the original inequality gives:

$$-3(1) + 1 \overset{?}{<} 4(1) - 5 + 2(1)$$
$$-2 < 1 \quad \text{is true; it checks}$$

- Choose a simple number that makes the final inequality '$x > \frac{2}{3}$' false: choose, say, $x = 0$. Substitution into the original inequality gives:

$$-3(0) + 1 \overset{?}{<} 4(0) - 5 + 2(0)$$
$$1 < -5 \quad \text{is false; it checks}$$

EXERCISE

9. Do a spot-check for the sentence '$-3x+1 < 4x-5+2x$' that is different from the one done above. That is, choose a number (different from 1) that makes '$x > \frac{2}{3}$' true, and substitute it into the original inequality. Then, choose a number (different from 0) that makes '$x > \frac{2}{3}$' false, and substitute it into the original inequality.
10. Solve the inequality, show the solution set on a number line, and spot-check: $5-2x+x \ge 4x-1$.

solving inequalities involving fractions or decimals

If an inequality involves fractions or decimals, it is usually easiest to clear them in the first step, and then proceed as usual. The spot-check for the next example is left as an exercise.

EXAMPLE

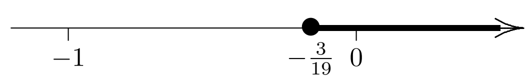
a linear inequality involving fractions

Solve: $\frac{1}{2}(1-5x) \le \frac{2}{3}x+1$. Shade the solution set on a number line. Spot-check your answer.

Solution: Note that 6 is the least common multiple of 2 and 3.

$\frac{1}{2}(1-5x) \le \frac{2}{3}x+1$	(original inequality)
$6(\frac{1}{2}(1-5x)) \le 6(\frac{2}{3}x+1)$	(multiply both sides by 6)
$3(1-5x) \le 4x+6$	(multiply out)
$3-15x \le 4x+6$	(simplify)
$3-19x \le 6$	(subtract $4x$ from both sides)
$-19x \le 3$	(subtract 3 from both sides)
$x \ge \frac{3}{-19}$	(divide both sides by -19; change direction of inequality)
$x \ge -\frac{3}{19}$	(simplify)

Here's the solution set:

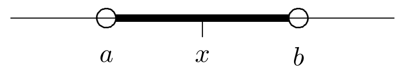


EXERCISE

11. Do a spot-check for the previous example.
12. Solve the inequality: $\frac{1}{3}x-5 > 3-\frac{2}{5}(x-1)$. Shade the solution set on a number line. Spot-check your answer.
13. Solve the inequality: $0.1x-5.3 < 1.04+x$. Shade the solution set on a number line. Spot-check your answer.

a common phrase:
'x is between a and b'

A common situation that arises in mathematics is the need to talk about values of x between a and b:



In this case, two things are true: x is greater than a, and x is less than b.

There is a common shorthand to talk about the values of x between a and b, which is '$a < x < b$'. The sentence '$a < x < b$' is called a *compound inequality*: the word 'compound' means 'to combine so as to form a whole'. Indeed, the compound inequality '$a < x < b$' is equivalent to two inequalities, put together with the mathematical word 'and':

DEFINITION
the compound inequality
'$a < x < b$'

For all real numbers a, x, and b,

$$a < x < b \quad \iff \quad \big(a < x \ \text{ and } \ x < b\big)\,.$$

Rewriting '$a < x$' as '$x > a$' gives the equivalent statement

$$a < x < b \quad \iff \quad \big(x > a \ \text{ and } \ x < b\big)\,.$$

'$a < x < b$'
represents
a whole family
of sentences

Recall that normal mathematical conventions dictate that letters near the beginning of the alphabet represent constants, and letters near the end of the alphabet represent real number variables. Thus, the single sentence '$a < x < b$' represents an entire family of sentences, where x varies within the sentence; a and b are held constant within a given sentence, but vary from sentence to sentence. Here are some members of the family represented by '$a < x < b$':

$$-1 < x < 3$$
$$\frac{1}{2} < x < \frac{2}{3}$$
$$-0.7 < x < \sqrt{2}$$

other
compound inequalities
of the same type

Here are other compound inequalities of the same type, with the values of x shaded that make them true (assuming that $a < b$):

$$a \le x \le b \quad \iff \quad \big(a \le x \ \text{ and } \ x \le b\big)$$

$$a < x \le b \quad \iff \quad \big(a < x \ \text{ and } \ x \le b\big)$$

$$a \le x < b \quad \iff \quad \big(a \le x \ \text{ and } \ x < b\big)$$

When is a compound inequality true?

In order for the mathematical sentence 'A and B' to be true, both A and B must be true. That is, an 'and' sentence is true only when both subsentences are true.

Be careful!

Thus, in order for '$a < x < b$' to be true, both '$a < x$' and '$x < b$' must be true. This happens only when a is less than b.

Be careful about this! It is common for beginning students of mathematics to write down sentences of this type that are always false!

For example, '$3 < x < 2$' is a compound inequality that is false for *all* values of x:

$$3 < x < 2 \iff \big(x > 3 \ \text{ and } \ x < 2\big)$$

There are no values of x that are greater than 3, and at the same time less than 2.

Don't put the inequalities in this direction!

Similarly, you must avoid situations like this:

$$\begin{aligned} 2 > x > 3 &\iff \big(2 > x \ \text{ and } \ x > 3\big) \\ &\iff \big(x < 2 \ \text{ and } \ x > 3\big) \end{aligned}$$

There are no values of x that are simultaneously less than 2, and greater than 3.

when working with '$a < x < b$':

make sure that a is less than b;

only use the symbols '$<$' and '$\le$'

When you work with sentences of the form '$a < x < b$', you usually want them to be *true* for certain values of x. Thus, always make sure that a is less than b, and only use the inequality symbols '$<$' and '$\le$'.

But what about the sentence '$3 > x > 2$'?

Answer: It is very unconventional! Avoid it!

But what about the sentence '$3 > x > 2$'?

$$\begin{aligned} 3 > x > 2 &\iff \big(3 > x \ \text{ and } \ x > 2\big) \\ &\iff \big(x < 3 \ \text{ and } \ x > 2\big) \\ &\iff \big(x > 2 \ \text{ and } \ x < 3\big) \\ &\iff \big(2 < x \ \text{ and } \ x < 3\big) \\ &\iff 2 < x < 3 \end{aligned}$$

As the sequence of equivalences shows, reading the sentence '$3 > x > 2$' from right-to-left instead of left-to-right gives the equivalent sentence '$2 < x < 3$', which is true for values of x between 2 and 3. However, it is *very unconventional* to write the sentence in the form '$3 > x > 2$'. It will cause mathematicians to look at you with a squinted eye, thinking 'Why did you write it this way?' So—stick to the normal conventions, and write the sentence as '$2 < x < 3$'.

mixing the directions is even worse:
$2 < x > 3$
Yuck!!

Here's a situation that is even worse. NEVER write something like '$2 < x > 3$', where the directions of the inequality symbols are mixed.

Why not? Well, here's what this means:

$$\begin{aligned} 2 < x > 3 \quad &\iff \quad \big(2 < x \ \text{ and } \ x > 3\big) \\ &\iff \quad \big(x > 2 \ \text{ and } \ x > 3\big) \\ &\iff \quad x > 3 \end{aligned}$$

The sentence '$2 < x > 3$' is equivalent to the simple inequality '$x > 3$'. There is *absolutely no reason* to write it in the more complicated way, and it would be extremely poor mathematical style to do so.

EXERCISES

14. On a number line, shade the value(s) of x that make each inequality true. If the sentence is always false, so state.

    If the sentence is written in an unconventional way, then shade the values of x that make it true, but also rewrite the sentence in the more conventional way.

    (a) $-1 < x < 2$
    (b) $-1 \le x < 2$
    (c) $-1 \le x \le 2$
    (d) $-1 < x \le 2$
    (e) $-1 > x > 2$
    (f) $2 > x > -1$
    (g) $2 \ge x > -1$
    (h) $2 < x < -1$

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: SOLVING LINEAR INEQUALITIES IN ONE VARIABLE

1. A linear inequality in x *must* have an x term.

2. (a) $2x - 5 \le 0$ is a linear inequality in x; it is in standard form

(b) $1 + 2x > 6x + \frac{1}{2}$ is a linear inequality in x

(c) $x^2 - x \ge 3$ is **not** a linear inequality in one variable; no x^2 term is allowed

(d) $0.4t - 7 < 2t$ is a linear inequality in t

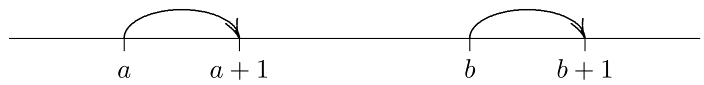
(e) $\dfrac{1}{t} + t \ge 0$ is **not** a linear inequality in one variable; no $\frac{1}{t}$ term is allowed

(f) $5.4(x - \frac{1}{3}) \le x - 0.2(1 + 8x)$ is a linear inequality in x

(g) $3x < 2y + 1$ is **not** a linear inequality in one variable; it uses two variables

3. (a) $a < b$

(b) $a + 1 < b + 1$



(c) $a - 1 < b - 1$

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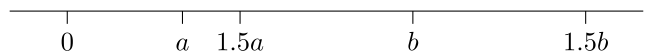
4. (a) $a < b$

(b) $\frac{1}{2}a < \frac{1}{2}b$

(c) $1.5a < 1.5b$



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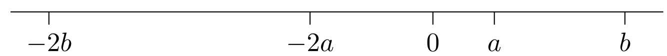
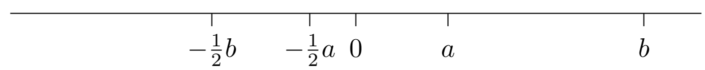
5. (a) $a < b$

(b) $-2a > -2b$



(c) $-\frac{1}{2}a > -\frac{1}{2}b$



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6. For all real numbers a, b and c,

$$a \le b \iff a + c \le b + c\,.$$

If $c > 0$, then

$$a \le b \iff ac \le bc\,.$$

If $c < 0$, then

$$a \le b \iff ac \ge bc\,.$$

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7. You can add (or subtract) the same number to (or from) both sides of an inequality, and this won't change the truth of the inequality.

8. If you multiply (or divide) both sides of an inequality by the same *negative* number, then you must change the direction of the inequality symbol in order to preserve the truth of the inequality.

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9. Choose a simple number that makes '$x > \frac{2}{3}$' true: choose, say, $x = 2$. Substitution into the original inequality gives:

$$-3(2) + 1 \overset{?}{<} 4(2) - 5 + 2(2)$$
$$-5 < 7 \quad \text{is true; it checks}$$

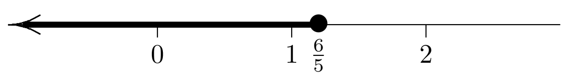
Next, choose a simple number that makes '$x > \frac{2}{3}$' false: choose, say, $x = -1$. Substitution into the original inequality gives:

$$-3(-1) + 1 \stackrel{?}{<} 4(-1) - 5 + 2(-1)$$
$$4 < -11 \quad \text{is false; it checks}$$

10.

$$\begin{aligned} &5 - 2x + x \ge 4x - 1 && \text{(original inequality)} \\ &5 - x \ge 4x - 1 && \text{(simplify)} \\ &5 - 5x \ge -1 && \text{(subtract } 4x \text{ from both sides)} \\ &-5x \ge -6 && \text{(subtract 5 from both sides)} \\ &x \le \frac{-6}{-5} && \text{(divide both sides by } -5\text{;} \\ & && \text{change direction of inequality)} \\ &x \le \frac{6}{5} && \text{(simplify)} \end{aligned}$$

Here's the solution set:



Here's the spot-check:
Choose $x = 1$:

$$5 - 2(1) + 1 \stackrel{?}{\ge} 4(1) - 1$$
$$4 \ge 3 \quad \text{is true; it checks}$$

Choose $x = 2$:

$$5 - 2(2) + 2 \stackrel{?}{\ge} 4(2) - 1$$
$$3 \ge 7 \quad \text{is false; it checks}$$

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11. original inequality: $\frac{1}{2}(1 - 5x) \le \frac{2}{3}x + 1$
final inequality: $x \ge \frac{3}{-19}$
Choose $x = 0$:

$$\frac{1}{2}(1 - 5(0)) \stackrel{?}{\le} \frac{2}{3}(0) + 1$$
$$\frac{1}{2} \le 1 \quad \text{is true; it checks}$$

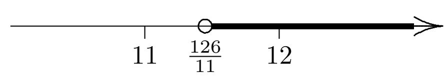
Choose $x = -1$:

$$\frac{1}{2}(1 - 5(-1)) \stackrel{?}{\le} \frac{2}{3}(-1) + 1$$
$$3 \le \frac{1}{3} \quad \text{is false; it checks}$$

12.

$\frac{1}{3}x - 5 > 3 - \frac{2}{5}(x-1)$	(original inequality)
$15(\frac{1}{3}x - 5) > 15(3 - \frac{2}{5}(x-1))$	(multiply both sides by 15)
$5x - 75 > 45 - 6(x-1)$	(simplify)
$5x - 75 > 45 - 6x + 6$	(simplify)
$5x - 75 > 51 - 6x$	(simplify)
$11x - 75 > 51$	(add $6x$ to both sides)
$11x > 126$	(add 75 to both sides)
$x > \frac{126}{11}$	(divide both sides by 11)

Note that $\frac{126}{11} \approx 11.5$. Here's the solution set:



Here's the spot-check:
Choose $x = 12$; use your calculator:

$$\frac{1}{3}(12) - 5 \overset{?}{>} 3 - \frac{2}{5}(12-1)$$
$$-1 > -1.4 \quad \text{is true; it checks}$$

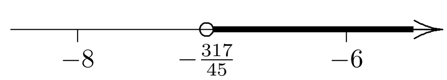
Choose $x = 11$; use your calculator:

$$\frac{1}{3}(11) - 5 \overset{?}{>} 3 - \frac{2}{5}(11-1)$$
$$-1.\overline{3} > -1 \quad \text{is false; it checks}$$

13.

$0.1x - 5.3 < 1.04 + x$	(original inequality)
$100(0.1x - 5.3) < 100(1.04 + x)$	(multiply both sides by 100)
$10x - 530 < 104 + 100x$	(simplify)
$-90x - 530 < 104$	(subtract $100x$ from both sides)
$-90x < 634$	(add 530 to both sides)
$x > \frac{634}{-90}$	(divide both sides by -90 ; change direction of inequality)
$x > -\frac{317}{45}$	(simplify)

Note that $-\frac{317}{45} \approx -7.04$. Here's the solution set:



Here's the spot-check:
Choose $x = -7$; use your calculator:

$$0.1(-7) - 5.3 \overset{?}{<} 1.04 + (-7)$$
$$-6 < -5.96 \quad \text{is true; it checks}$$

Choose $x = -8$; use your calculator:

$$0.1(-8) - 5.3 \overset{?}{<} 1.04 + (-8)$$
$$-6.1 < -6.96 \quad \text{is false; it checks}$$

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14. (a) $-1 < x < 2$:

(b) $-1 \le x < 2$:

(c) $-1 \le x \le 2$:

(d) $-1 < x \le 2$:

(e) $-1 > x > 2$: always false

(f) $2 > x > -1$; rewrite as $-1 < x < 2$:

(g) $2 \ge x > -1$: rewrite as $-1 < x \le 2$:

(h) $2 < x < -1$: always false

