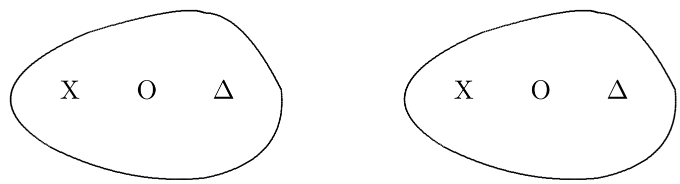
2. BRUSHING UP ON BASIC ARITHMETIC SKILLS

too much reliance on calculators

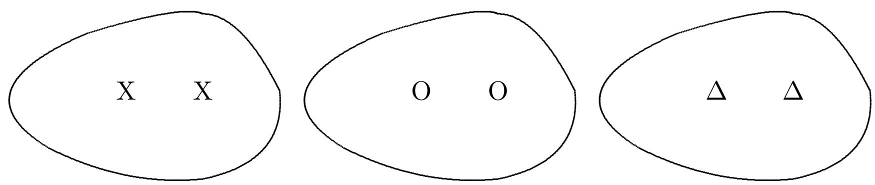
With the trend towards more and earlier calculator usage, some people have lost a comfort with basic arithmetic operations like $5 \cdot 7 = 35$ and $8 + 6 = 14$. It is a waste of valuable time to use your calculator for problems such as these. In this section, your basic arithmetic skills are brought "up to speed" so you won't be wasting mental energy on arithmetic and will be able to concentrate on higher-level ideas.

multiplication concept

Consider the multiplication problem $2 \cdot 3$. Recall that the centered dot is used to denote multiplication. One interpretation of $2 \cdot 3$ is two piles with three in each pile:



Notice that these six objects can be re-arranged into three piles with two in each pile, as illustrated below:



Thus, $2 \cdot 3 = 3 \cdot 2$. In general, $xy = yx$ for all numbers x and y. That is, you can *commute* (change the places of) the numbers in a multiplication problem, and this does not affect the result. This fact is called the *Commutative Property of Multiplication*.

EXERCISES
practice with the 'commutative' concept

1. What is the *Commutative Property of Multiplication*?
2. What do you suppose is the *Commutative Property of Addition*?
3. Do you think there is a *Commutative Property of Subtraction*? Answer yes or no, and give a reason for your answer.

the multiplication tables

You should be comfortable and efficient with your multiplication tables through 10. Here's a quick review, noting some special properties:

multiplying by 0

Any number times 0 is 0:

$0 \cdot 3 = 0$ $\qquad$ $4 \cdot 0 = 0$ $\qquad$ $0 \cdot 0 = 0$ $\qquad$ $0 \cdot \frac{1}{2} = 0$ etc.

That is $x \cdot 0 = 0 \cdot x = 0$ for all numbers x.

multiplying by 1

Any number times 1 is the original number:

$1 \cdot 3 = 3$ $\qquad$ $4 \cdot 1 = 4$ $\qquad$ $1 \cdot 1 = 1$ $\qquad$ $1 \cdot \frac{1}{2} = \frac{1}{2}$ etc.

That is, $x \cdot 1 = 1 \cdot x = x$ for all numbers x. Multiplying a number by 1 preserves the identity of (i.e., doesn't change) the original number. Thus, 1 is called the *multiplicative identity*.

EXERCISES
practice with the 'identity' concept

4. Why is the number 1 called the *multiplicative identity*?
5. Is there an *additive identity*? That is, is there a number with the property that when you add it to any number, it preserves the identity of (i.e., doesn't change) that number?

multiplying by 2

Here are the multiplication by 2 facts that you should know:

$2 \cdot 0 = 0$	$2 \cdot 1 = 2$	$2 \cdot 2 = 4$	$2 \cdot 3 = 6$
$2 \cdot 4 = 8$	$2 \cdot 5 = 10$	$2 \cdot 6 = 12$	$2 \cdot 7 = 14$
$2 \cdot 8 = 16$	$2 \cdot 9 = 18$	$2 \cdot 10 = 20$	

The numbers 0, 2, 4, 6, 8, 10, 12, ... are called *even numbers*. Even numbers always end in 0, 2, 4, 6, or 8. Even numbers can always be divided into two equal (even) piles.

divisibility

The numbers 0, 2, 4, 6, ... are also said to be *divisible by* 2. 'Divisible by 2' means that 2 goes into the number evenly. The phrases 'even' and 'divisible by 2' are interchangeable.

As a second example to illustrate the divisibility concept, consider the numbers 0, 3, 6, 9, 12, 15, ... These numbers are divisible by 3. 'Divisible by 3' means that 3 goes into the number evenly. A number that is divisible by 3 can be divided into 3 equal piles.

divisibility test

A 'divisibility test' is a shortcut to decide if a number is divisible by a given number. Several popular divisibility tests are introduced in this section. Here's the first one:

divisibility by 2

If a number ends in 0, 2, 4, 6, or 8, then the number is divisible by 2.

Also, if a number is divisible by 2, then it ends in 0, 2, 4, 6, or 8.

EXAMPLE
divisibility by 2

For example, 137,396 and 9,180 are both divisible by 2, but 3,981 is not divisible by 2.

EXERCISES
practice with the 'divisibility' concept

6. Suppose that abc is a 3-digit number. If it is divisible by 2, then what (if anything) can be said about a, b, and c?
7. Suppose that abc is a 3-digit number. If it is not divisible by 2, then what (if anything) can be said about a, b, and c?
8. Give an example of a 5-digit number that is divisible by 2.
9. Give an example of a 4-digit even number.
10. Give an example of a 4-digit number that is not divisible by 2.
11. Give an example of a 5-digit number that is not even.

multiplying by 3

Here are the multiplication by 3 facts that you should know:

$3 \cdot 0 = 0$	$3 \cdot 1 = 3$	$3 \cdot 2 = 6$	$3 \cdot 3 = 9$
$3 \cdot 4 = 12$	$3 \cdot 5 = 15$	$3 \cdot 6 = 18$	$3 \cdot 7 = 21$
$3 \cdot 8 = 24$	$3 \cdot 9 = 27$	$3 \cdot 10 = 30$	

fun fact about multiplication by 3

Here's a fun fact about multiplication by 3. Take any number that is divisible by 3, like 186. Add up the digits in the number: $1+8+6=15$. Notice that 15 is also divisible by 3. Add up the digits again: $1+5=6$. Again, 6 is divisible by 3.

Indeed, if a number is divisible by 3, then the sum of its digits is also divisible by 3. If a number isn't divisible by 3, then the sum of its digits isn't divisible by 3. This idea leads to the following test for divisibility by 3:

divisibility by 3

To decide if a number is divisible by 3, add up the digits in the number. Continue this process of adding the digits until you get a manageable number. (If you want, keep going until you get a single-digit number.) If this final number is divisible by 3, then the number you started with is also divisible by 3. If this final number is not divisible by 3, then the number you started with is not divisible by 3.

★

proof of the divisibility by 3 *test for three-digit numbers*

Material that is labeled with a ★ is included for the benefit of more advanced readers, and can be skipped without any loss of continuity.

Here's a proof of the 'divisibility by three' test for three-digit numbers. The idea illustrated here extends easily to all base ten numbers.

Let N represent a three-digit number, where a is the digit in the hundreds place, b is the digit in the tens place, and c is the digit in the ones place. Thus,

$$\begin{aligned} N &= 100a + 10b + c \\ &= 99a + a + 9b + b + c \\ &= (99a + 9b) + (a + b + c) \\ &= 3(33a + 3b) + (a + b + c) \end{aligned}$$

If N is divisible by 3, then the sum of its digits, $a+b+c$, must also be divisible by 3. If the sum $a+b+c$ is divisible by 3, then so is N. Together, N is divisible by 3 if and only if the sum of its digits is divisible by 3.

EXAMPLE

divisibility by 3

For example, suppose you want to know if 9,325 is divisible by 3. With a calculator, you could divide 9,325 by 3 to see if it goes in evenly. In the absence of a calculator, and yet armed with the divisibility by 3 test, you can do this: $9+3+2+5=19$; $1+9=10$. Clearly, 10 is not divisible by 3, so the original number 9,325 is not divisible by 3.

no special remarks about 4*,* 6*,* 7*, and* 8

There are no special remarks about multiplication by 4, 6, 7, or 8, so these multiplication facts are postponed until the end of the section.

multiplying by 5

Here are the multiplication by 5 facts that you should know:

$5\cdot 0=0$	$5\cdot 1=5$	$5\cdot 2=10$	$5\cdot 3=15$
$5\cdot 4=20$	$5\cdot 5=25$	$5\cdot 6=30$	$5\cdot 7=35$
$5\cdot 8=40$	$5\cdot 9=45$	$5\cdot 10=50$	

Notice that all these numbers end in 0 and 5. This leads us to:

divisibility by 5

If a number ends in 0 or 5, then it is divisible by 5.
Also, if a number is divisible by 5, then it ends in 0 or 5.

a compact statement of divisibility by 5

The two sentences in the box above can be stated extremely compactly using the language of mathematics, like this:

A number is divisible by 5 if and only if the number ends in 0 or 5 .

The words ‘ if and only if ’ will be discussed in detail in a later section. They are introduced here only to begin to illustrate the efficiency of the mathematical language.

multiplying by 9

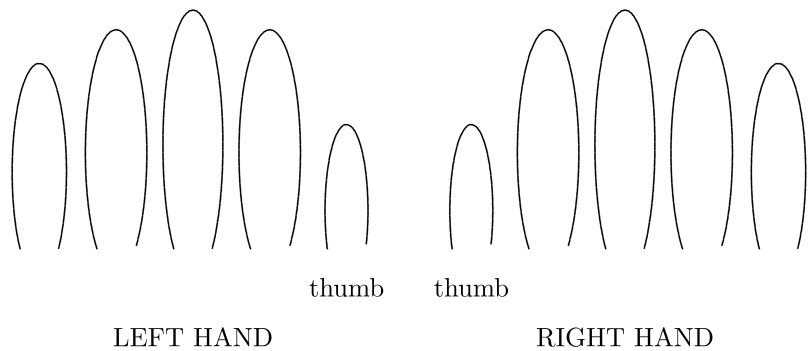
Here are the multiplication by 9 facts that you should know:

$9 \cdot 0 = 0$	$9 \cdot 1 = 9$	$9 \cdot 2 = 18$	$9 \cdot 3 = 27$
$9 \cdot 4 = 36$	$9 \cdot 5 = 45$	$9 \cdot 6 = 54$	$9 \cdot 7 = 63$
$9 \cdot 8 = 72$	$9 \cdot 9 = 81$	$9 \cdot 10 = 90$	

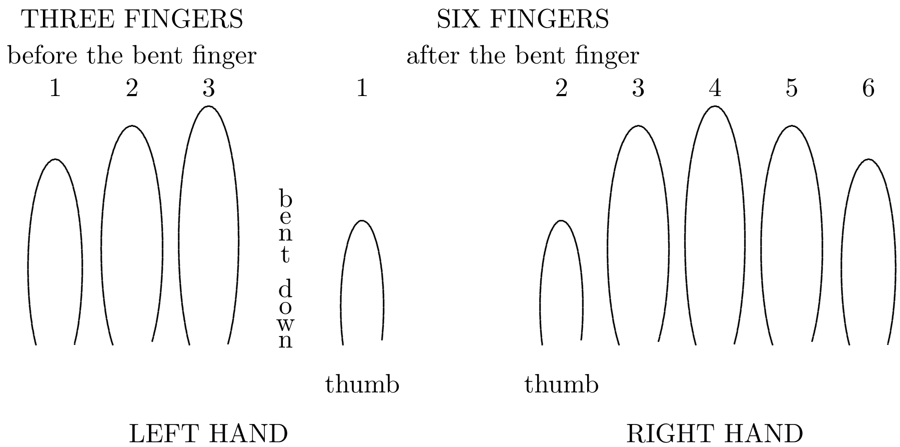
clever finger trick

There’s a clever ‘finger trick’ to help you with multiplication by 9 . This trick is illustrated by finding $9 \cdot 4$:

First, hold your ten fingers in front of you, as shown below. (Note: Drawing capabilities are severely limited within TEX , which is the technical word processor used to typeset mathematics. So, please use your imagination to help the images below!)



Since you want 9 times 4 , bend your fourth finger from the left down, like this:



There are 3 fingers before the bent finger, and 6 fingers after, so $4 \cdot 9 = 36$.

You should check that this technique always works. After you do it a few times, you won’t actually have to hold your hands in front of you—you’ll be able to visualize them in your head.

multiplying by 10

Here are the multiplication by 10 facts that you should know:

$10 \cdot 0 = 0$	$10 \cdot 1 = 10$	$10 \cdot 2 = 20$	$10 \cdot 3 = 30$
$10 \cdot 4 = 40$	$10 \cdot 5 = 50$	$10 \cdot 6 = 60$	$10 \cdot 7 = 70$
$10 \cdot 8 = 80$	$10 \cdot 9 = 90$	$10 \cdot 10 = 100$	

Notice that a number is divisible by 10 if and only if it ends with a 0.

the remaining multiplication facts

Here are the remaining multiplication by facts that you should know:

$4 \cdot 0 = 0$	$6 \cdot 0 = 0$	$7 \cdot 0 = 0$	$8 \cdot 0 = 0$
$4 \cdot 1 = 4$	$6 \cdot 1 = 6$	$7 \cdot 1 = 7$	$8 \cdot 1 = 8$
$4 \cdot 2 = 8$	$6 \cdot 2 = 12$	$7 \cdot 2 = 14$	$8 \cdot 2 = 16$
$4 \cdot 3 = 12$	$6 \cdot 3 = 18$	$7 \cdot 3 = 21$	$8 \cdot 3 = 24$
$4 \cdot 4 = 16$	$6 \cdot 4 = 24$	$7 \cdot 4 = 28$	$8 \cdot 4 = 32$
$4 \cdot 5 = 20$	$6 \cdot 5 = 30$	$7 \cdot 5 = 35$	$8 \cdot 5 = 40$
$4 \cdot 6 = 24$	$6 \cdot 6 = 36$	$7 \cdot 6 = 42$	$8 \cdot 6 = 48$
$4 \cdot 7 = 28$	$6 \cdot 7 = 42$	$7 \cdot 7 = 49$	$8 \cdot 7 = 56$
$4 \cdot 8 = 32$	$6 \cdot 8 = 48$	$7 \cdot 8 = 56$	$8 \cdot 8 = 64$
$4 \cdot 9 = 36$	$6 \cdot 9 = 54$	$7 \cdot 9 = 63$	$8 \cdot 9 = 72$
$4 \cdot 10 = 40$	$6 \cdot 10 = 60$	$7 \cdot 10 = 70$	$8 \cdot 10 = 80$

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTIONS TO EXERCISES: BRUSHING UP ON BASIC ARITHMETIC SKILLS

1. $xy = yx$ for all numbers x and y
2. $x + y = y + x$ for all numbers x and y
3. $3 - 2$ is not equal to $2 - 3$; there is no Commutative Property of Subtraction
4. Multiplying a number by 1 preserves the identity of (i.e., doesn't change) the number.
5. 0 is the additive identity: $0 + x = x + 0 = x$
6. c must be 0, 2, 4, 6 or 8
7. c must be 1, 3, 5, 7 or 9
8. 32,584 (there are many correct examples)
9. 3,098 (there are many correct examples)
10. 9,781 (there are many correct examples)
11. 23,647 (there are many correct examples)