35. SOLVING ABSOLUTE VALUE EQUATIONS

solving equations involving absolute value

This section presents the tool needed to solve absolute value equations like these:

$$|x| = 5$$
$$|2 - 3x| = 7$$
$$5 - 2|3 - 4x| = -7$$

Each of these equations has only a single set of absolute value symbols, and has a *variable* inside the absolute value. Solving sentences like these is easy, if you remember the critical fact that

absolute value gives distance from 0 .

Keep this in mind as you read the following theorem:

THEOREM

tool for solving absolute value equations

Let $x \in \mathbb{R}$, and let $k \ge 0$. Then,

$$|x| = k \quad \iff \quad x = \pm k \ .$$

$|x| = k$ is an entire class of sentences

Recall first that normal mathematical conventions dictate that ' $|x| = k$ ' represents an entire class of sentences, including $|x| = 2$, $|x| = 5.7$, and $|x| = \frac{1}{3}$. The variable k changes from sentence to sentence, but is constant within a given sentence.

EXERCISES

1. a. Give three sentences of the form ' $|x| = k$ ' where $k \ge 0$. Use examples different from those given above.
   b. Give three sentences of the form ' $x = \pm k$ ' where $k \ge 0$.

translating the theorem:

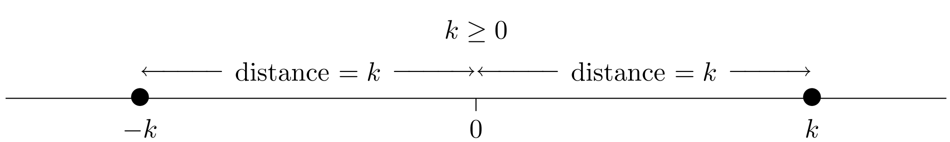
thought process for solving sentences like $|x| = k$

When you see a sentence of the form ' $|x| = k$ ', here's what you should do:

- Check that k is a nonnegative number.
- The symbol $|x|$ represents the distance between x and 0 .
- Thus, you want the numbers x , whose distance from 0 is k .

$$\overbrace{|x|}^{\text{want numbers } x\text{, whose distance from 0}} \qquad \overbrace{=}^{\text{is}} \quad \overbrace{k}^{k}$$

- You can walk from 0 in two directions: to the right, or to the left. Thus, there are two numbers whose distance from 0 is the nonnegative number k : k and $-k$.



- Thus, $|x| = k$ is equivalent to $x = \pm k$.

filling in some blanks to help your thought process

When you see a sentence like ‘ $|x| = 7$ ’, you thought process should be like filling in the following blanks:

We want the numbers ______ , whose ______ from ______ is ______ . Thus, we want ______ to be ______ or ______ .

The correctly-filled-in blanks are:

We want the numbers $\underline{x}$, whose <u>distance</u> from $\underline{0}$ is $\underline{7}$. Thus, we want $\underline{x}$ to be $\underline{7}$ or $\underline{-7}$.

EXERCISES

2. Fill in the blanks:
   a. When you look at the sentence ‘ $|x| = 5$ ’, you should think: We want the numbers ______ , whose ______ from ______ is ______ . Thus, we want ______ to be ______ or ______ .
   b. When you look at the sentence ‘ $|z| = \frac{1}{5}$ ’, you should think: We want the numbers ______ , whose ______ from ______ is ______ . Thus, we want ______ to be ______ or ______ .
   c. When you look at the sentence ‘ $|x| = k$ ’ (with $k \geq 0$), you should think: We want the numbers ______ , whose ______ from ______ is ______ . Thus, we want ______ to be ______ or ______ .
3. Give a sentence, not using absolute value symbols, that is equivalent to:
   a. $|x| = 3$
   b. $|t| = 4.2$
4. Give a sentence, using absolute value symbols, that is equivalent to:
   a. $x = \pm 7$
   b. $t = \frac{1}{3}$ or $t = -\frac{1}{3}$
5. Give a precise mathematical statement of the tool that says that a sentence like ‘ $|x| = 5$ ’ can be transformed to the equivalent sentence ‘ $x = \pm 5$ ’.
6. Is the sentence ‘ $|x| = -6$ ’ of the form described in the previous theorem? Why or why not?
7. Can the sentence ‘ $|x| - 5 = 7$ ’ be transformed to a sentence of the form described in the previous theorem? If so, what is the equivalent sentence?

EXAMPLE

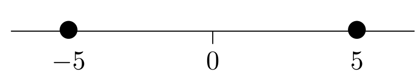
solving a sentence of the form $|x| = k$

Example: Solve: $|x| = 5$

Solution:

$$|x| = 5$$
$$x = \pm 5$$

The solution set is shaded below:



solving more complicated sentences of the form $|x| = k$;

x can be ANYTHING!

The power of the tool

$$|x| = k \iff x = \pm k$$

goes way beyond solving simple sentences like '$|x| = 5$'. Since x can be *any* real number, you should think of x as merely representing *the stuff inside the absolute value symbols*. Thus, you could think of rewriting the tool as:

$$|\text{stuff}| = k \iff \text{stuff} = \pm k$$

Thus, we have all the following equivalences:

$$|\overbrace{2-3x}^{\text{stuff}}| = 7 \iff \overbrace{2-3x}^{\text{stuff}} = \pm 7$$

$$|\overbrace{5x-1}^{\text{stuff}}| = 8 \iff \overbrace{5x-1}^{\text{stuff}} = \pm 8$$

$$|\overbrace{x^2-3x+4}^{\text{stuff}}| = \frac{1}{5} \iff \overbrace{x^2-3x+4}^{\text{stuff}} = \pm\frac{1}{5}$$

and so on!

EXERCISES

8. For each of the following, write an equivalent sentence that does not use absolute value symbols. Do *not* solve the resulting sentences.
   a. $|1 + 2x| = 3$
   b. $|7x - \frac{1}{2}| = 5$
   c. $|x^2 - 8| = 0.4$

EXAMPLE

Example: Solve: $|2 - 3x| = 7$

Solution: Be sure to write a nice, clean list of equivalent sentences.

$\lvert 2-3x\rvert = 7$	original sentence
$2 - 3x = \pm 7$	$\lvert x\rvert = k \iff x = \pm k$
$-3x = \pm 7 - 2$	subtract 2 from both sides
$x = \dfrac{\pm 7 - 2}{-3}$	divide both sides by -3
$x = \dfrac{7-2}{-3}$ or $x = \dfrac{-7-2}{-3}$	$x = \pm k \iff (x = k \text{ or } x = -k)$
$x = -\dfrac{5}{3}$ or $x = 3$	arithmetic

Checking gives:

$$|2 - 3(-\frac{5}{3})| \stackrel{?}{=} 7$$
$$|2 + 5| \stackrel{?}{=} 7$$
$$7 = 7$$

$$|2 - 3(3)| \stackrel{?}{=} 7$$
$$| - 7| \stackrel{?}{=} 7$$
$$7 = 7$$

EXAMPLE

an alternate approach to the previous example

Example: Here, the previous example is repeated, except this time without using the '$\pm$' notation. Use whichever approach is more comfortable for you.

Solve: $|2 - 3x| = 7$

Solution:

$$|2 - 3x| = 7$$
$$2 - 3x = 7 \quad \text{or} \quad 2 - 3x = -7$$
$$-3x = 5 \quad \text{or} \quad -3x = -9$$
$$x = -\frac{5}{3} \quad \text{or} \quad x = 3$$

an alternate form for the check

Here is an alternate way to write down the check, which is a bit more compact. This method works nicely when the the original equation has a constant on one side. Notice that the value of x is substituted into the side of the equation containing the variable, and it is shown to equal the desired constant.

$$|2 - 3(-\frac{5}{3})| = |2 + 5| = 7$$
$$|2 - 3(3)| = | - 7| = 7$$

EXERCISES

9. Solve. Write a nice, clean list of equivalent sentences. Check your solutions.
   a. $|1 + 2x| = 3$
   b. $|7x - 5| = 2$
   c. $|4 - 9x| = 1$

What happens if k is negative in the sentence '$|x| = k$'?

What about a sentence like '$|x| = -5$', where the absolute value is equal to a negative number? Notice that this situation is not covered in the previous theorem, since the sentence '$|x| = k$' is only addressed with $k \ge 0$.

Recall that $|x| \ge 0$ for all real numbers x. Thus, the sentence '$|x| = -5$' is never true: the left-hand side is always greater than or equal to 0, and the right-hand side is less than 0:

$$\overbrace{|x|}^{\ge 0} = \overbrace{-5}^{<0}$$

Even when x is 5 or -5, the sentence is false, as shown below:

x	substitution into '$|x| = -5$'	true or false?
5	$|5| = -5$	false
-5	$|-5| = -5$	false

first step when analyzing $|x| = k$: check that $k \ge 0$

Whenever you're working with a sentence of the form '$|x| = k$', you must always check first that $k \ge 0$. If k is negative, you just stop and say that the sentence is always false. Here are some examples, which illustrate different ways that you can state your answer:

'$|x| = -3$' is always false.

'$|2x - 1| = -5$' is never true.

'$|3x - 5x^2 + 7| = -0.4$' has an empty solution set.

EXERCISES

10. Decide which of the following sentences are always false. Do NOT solve the sentences.
    a. $|x| = -9$
    b. $|x| = 0$
    c. $|3x - 5| = -4.7$
    d. $|1 - 4x| + 5 = 0$
    e. $-2|x^2 + 3x - 1| = 8$
    f. $|9x + 1| - 5 = -3$

EXAMPLE
putting a sentence in standard form first

Sometimes you need a few transformations to get an equivalent sentence in the form $|x| = k$, as the next example illustrates.

Solve: $5 - 2|3 - 4x| = -7$

$5 - 2\|3 - 4x\| = -7$	original sentence
$-2\|3 - 4x\| = -12$	subtract 5 from both sides
$\|3 - 4x\| = 6$	divide both sides by -2
$3 - 4x = 6$ or $3 - 4x = -6$	$\|x\| = k \iff x = \pm k$
$-4x = 3$ or $-4x = -9$	addition property of equality
$x = -\frac{3}{4}$ or $x = \frac{9}{4}$	multiplication property of equality

Checking:

$$5 - 2|3 - 4(-\frac{3}{4})| = 5 - 2|3 + 3| = 5 - 2|6| = 5 - 2(6) = 5 - 12 = -7$$
$$5 - 2|3 - 4(\frac{9}{4})| = 5 - 2|3 - 9| = 5 - 2|-6| = 5 - 2(6) = 5 - 12 = -7$$

EXERCISES

11. Solve and check each of the following equations. Be sure to write a nice, clean list of equivalent equations.
    a. $7 - 5|1 - 2x| = -3$
    b. $-3|2x - 1| - 5 = -4$
    c. $2|3x - 5| - 1 = 7$

EXERCISES
web practice

Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTION TO EXERCISES: SOLVING ABSOLUTE VALUE EQUATIONS

1. a. $|x| = 1$, $|x| = 7.2$, and $|x| = \frac{1}{2}$

b. $x = \pm 1$, $x = \pm 7.2$, and $x = \pm\frac{1}{2}$

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2. a. We want the numbers $\underline{x}$, whose <u>distance</u> from $\underline{0}$ is $\underline{5}$. Thus, we want $\underline{x}$ to be $\underline{5}$ or $\underline{-5}$.

b. We want the numbers $\underline{z}$, whose <u>distance</u> from $\underline{0}$ is $\underline{\frac{1}{5}}$. Thus,we want $\underline{z}$ to be $\underline{\frac{1}{5}}$ or $\underline{-\frac{1}{5}}$.

c. We want the numbers $\underline{x}$, whose <u>distance</u> from $\underline{0}$ is $\underline{k}$. Thus, we want $\underline{x}$ to be $\underline{k}$ or $\underline{-k}$.

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3. a. $|x| = 3$ is equivalent to $x = \pm 3$

b. $|t| = 4.2$ is equivalent to $t = \pm 4.2$

4. a. $x = \pm 7$ is equivalent to $|x| = 7$

'$t = \frac{1}{3}$ or $t = -\frac{1}{3}$' is equivalent to $|t| = \frac{1}{3}$

5. For all real numbers x, and for $k \geq 0$, $|x| = k$ is equivalent to $x = \pm k$.

6. The sentence '$|x| = -6$' is not of the form in the theorem, because -6 is a negative number.

7. The sentence '$|x| - 5 = 7$' can be transformed to '$|x| = 12$' by adding 5 to both sides.

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8. a. $|1 + 2x| = 3$ is equivalent to $1 + 2x = \pm 3$
b. $|7x - \frac{1}{2}| = 5$ is equivalent to $7x - \frac{1}{2} = \pm 5$
c. $|x^2 - 8| = 0.4$ is equivalent to $x^2 - 8 = \pm 0.4$

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9. a.

$$
\begin{gathered}
|1 + 2x| = 3 \\
1 + 2x = 3 \quad \text{or} \quad 1 + 2x = -3 \\
2x = 2 \quad \text{or} \quad 2x = -4 \\
x = 1 \quad \text{or} \quad x = -2
\end{gathered}
$$

Check: $|1 + 2(1)| = |1 + 2| = |3| = 3$;
$|1 + 2(-2)| = |1 - 4| = |-3| = 3$

b.

$$
\begin{gathered}
|7x - 5| = 2 \\
7x - 5 = 2 \quad \text{or} \quad 7x - 5 = -2 \\
7x = 7 \quad \text{or} \quad 7x = 3 \\
x = 1 \quad \text{or} \quad x = \frac{3}{7}
\end{gathered}
$$

Check: $|7(1) - 5| = |2| = 2$;
$|7(\frac{3}{7}) - 5| = |3 - 5| = |-2| = 2$

c.

$$
\begin{gathered}
|4 - 9x| = 1 \\
4 - 9x = 1 \quad \text{or} \quad 4 - 9x = -1 \\
-9x = -3 \quad \text{or} \quad -9x = -5 \\
x = \frac{1}{3} \quad \text{or} \quad x = \frac{5}{9}
\end{gathered}
$$

Check: $|4 - 9(\frac{1}{3})| = |4 - 3| = 1$; $|4 - 9(\frac{5}{9})| = |4 - 5| = |-1| = 1$

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10. (a), (c), (d), and (e)are always false; some explanations follow:
Note that $|1 - 4x| + 5 = 0$ is equivalent to $|1 - 4x| = -5$.
Note that $-2|x^2 + 3x - 1| = 8$ is equivalent to $|x^2 + 3x - 1| = -4$.
Note that $|9x + 1| - 5 = -3$ is equivalent to $|9x + 1| = 2$.

11. a.

$$7 - 5|1 - 2x| = -3$$
$$-5|1 - 2x| = -10$$
$$|1 - 2x| = 2$$
$$1 - 2x = 2 \quad \text{or} \quad 1 - 2x = -2$$
$$-2x = 1 \quad \text{or} \quad -2x = -3$$
$$x = -\frac{1}{2} \quad \text{or} \quad x = \frac{3}{2}$$

Check:

$$7 - 5|1 - 2(-\frac{1}{2})| = 7 - 5|1 + 1| = 7 - 5(2) = 7 - 10 = -3$$
$$7 - 5|1 - 2(\frac{3}{2})| = 7 - 5|1 - 3| = 7 - 5|-2| = 7 - 5(2) = 7 - 10 = -3$$

b.

$$-3|2x - 1| - 5 = -4$$
$$-3|2x - 1| = 1$$
$$|2x - 1| = -\frac{1}{3}$$

always false!

c.

$$2|3x - 5| - 1 = 7$$
$$2|3x - 5| = 8$$
$$|3x - 5| = 4$$
$$3x - 5 = 4 \quad \text{or} \quad 3x - 5 = -4$$
$$3x = 9 \quad \text{or} \quad 3x = 1$$
$$x = 3 \quad \text{or} \quad x = \frac{1}{3}$$

Check:

$$2|3(3) - 5| - 1 = 2|9 - 5| - 1 = 2|4| - 1 = 2(4) - 1 = 8 - 1 = 7$$
$$2|3(\frac{1}{3}) - 5| - 1 = 2|1 - 5| - 1 = 2|-4| - 1 = 2(4) - 1 = 8 - 1 = 7$$