34. INTRODUCTION TO ABSOLUTE VALUE

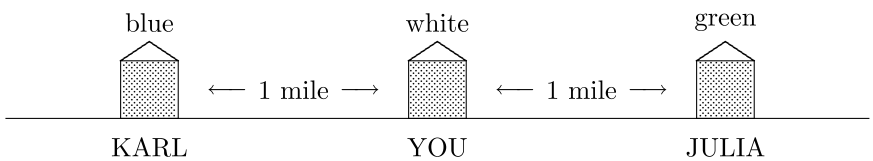
Karl and Julia live the same distance from you

Suppose you live in the big white house shown below. Julia lives in the green house to the right, and Karl lives in the blue house to the left.

Karl and Julia live in different places, but they both live the same distance from your home. To visit Julia, you would walk to the right. To visit Karl, you would walk to the left.

Suppose you were asked the question: "Who lives one mile from you on this street?" You would respond by saying: "Well—two people. Julia lives one mile this way, and Karl lives one mile that way."

This section explores the idea of *distance from home*, under the guise of a mathematical concept called *absolute value*.



distance from zero:

3 and −3 have the same distance from 0

Now, think of "home" as being position 0 on a number line. Instead of talking about *distance from home*, we'll talk about *distance from zero*.

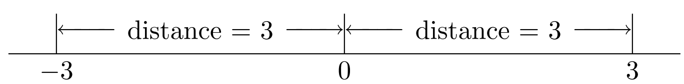
When you talk about distance from zero, it's critical to remember that you can walk away from 0 in two directions: to the right, or to the left.

Suppose you're asked the question: "What numbers are three units from 0 on a number line?"

The number 3 is three units from 0; it is 3 units to the right of 0.

Also, the number -3 is three units from 0; it is 3 units to the left of 0.

Opposites (like 3 and -3) always have the same distance from 0: one is to the right of zero, and one is to the left of zero.

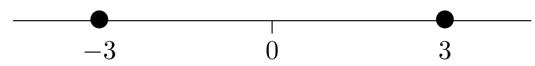


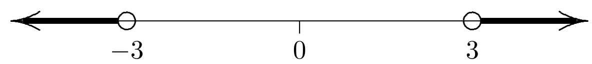
EXERCISES

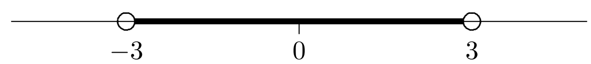
1. On a number line, shade the specified number(s):
   (a) all number(s) that are 2 units to the right of 0
   (b) all number(s) that are 2 units to the left of 0
   (c) all number(s) that are 2 units from 0
   (d) all positive number(s) that are 2 units from 0
   (e) all negative number(s) that are 2 units from 0
   (f) all number(s) that are more than 2 units to the right of 0
   (g) all number(s) that are more than 2 units to the left of 0
   (h) all number(s) that are more than 2 units from 0
   (i) all positive number(s) that are more than 2 units from 0
   (j) all negative number(s) that are more than 2 units from 0
   (k) all number(s) that are less than 2 units to the right of 0
   (l) all number(s) that are less than 2 units to the left of 0
   (m) all number(s) that are less than 2 units from 0 (include 0)
   (n) all positive number(s) that are less than 2 units from 0
   (o) all negative number(s) that are less than 2 units from 0

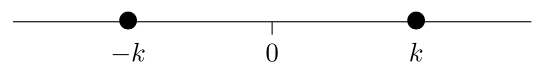
EXERCISES

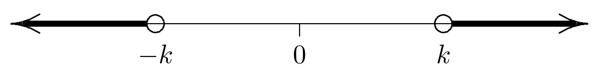
2. Describe the numbers shaded below, using phrases similar to those in the previous exercise. Assume that k is a positive number.

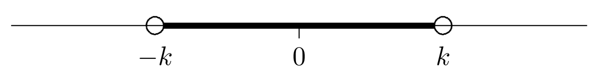
(a)


(b)


(c)


(d)


(e)


(f)


the notation '$|x|$' represents the distance between x and 0

The notation '$|x|$' (the number x inside vertical bars) is used to represent the distance between the number x and 0. (You'll be told how to read the symbol '$|x|$' in just a moment.)

When you see the vertical bars '$|stuff|$', your first thought should be **distance from** 0.

The symbol '$|3|$' represents the distance between 3 and 0. Thus, $|3| = 3$, since the number 3 is 3 units from zero.

The symbol '$|-3|$' represents the distance between -3 and 0. Thus, $|-3| = 3$, since the number -3 is 3 units from zero.

Notice that $|3|$ and $|-3|$ are both mathematical *expressions*—they are numbers.

EXERCISES

3. Fill in the blanks:
   (a) The symbol '$|4|$' represents the distance between _____ and _____. Thus, $|4| =$_____ , since the number _____ is _____ units from zero.
   (b) The symbol '$|-4|$' represents the distance between _____ and _____. Thus, $|-4| =$_____ , since the number _____ is _____ units from zero.

The expression '$|x|$' is called the *absolute value of* x. There are two common definitions of $|x|$: a geometric definition (which follows), and an algebraic definition. The algebraic definition is presented in a future section.

DEFINITION
the absolute value of x ;
geometric definition

Let x be a real number. Then,

$$|x| = \text{the distance between } x \text{ and } 0\,.$$

The symbol $|x|$ is read as *the absolute value of* x.

reading absolute values aloud

The symbol ‘ $|3|$ ’ is read as ‘the absolute value of 3 ’.

The symbol ‘ $|-3|$ ’ is read as ‘the absolute value of -3 ’.

The symbol ‘ $|2x-1|$ ’ is read as ‘the absolute value of two ex minus one’.

EXERCISES

4. State how you would read the symbols aloud.

   Then, classify as an expression or a sentence.

   Give the truth value of any sentence: always true (T), always false (F), or sometimes true/sometimes false (ST/SF).

   If a sentence is ST/SF, then give a value of x for which is is true, and a value of x for which it is false.

   (a) $|-7|$
   (b) $|1-3x|$
   (c) $|-3|>2$
   (d) $|-3|<2$
   (e) $|-3|=3$
   (f) $|3|=-3$
   (g) $|-3|=-3$
   (h) $|x|=3$
   (i) $|x|=-3$

opposites have the same distance from zero

Opposites have the same distance from zero, so opposites have the same absolute value:

$|3|=|-3|$;

$|4.2|=|-4.2|$;

in general, $|x|=|-x|$ for all real numbers x .

distance is always greater than or equal to zero

Whenever you report distance, you give a number that is greater than or equal to zero. You might say something like ‘this number is 2 units from 0’, but you would *never* say ‘this number is -2 units from 0’.

Since $|x|$ reports a distance, it follows that $|x|$ is always greater than or equal to zero. Thus, $|x|\ge 0$ for all real numbers x . Recall that ‘nonnegative’ means ‘greater than or equal to zero’. Thus, $|x|$ is a nonnegative quantity.

No matter *what* is inside the absolute value symbols, the resulting number is always nonnegative. For example, $|-x^2+3x-1|\ge 0$ for all real numbers x .

The observations in the preceding paragraphs are summarized as two important properties of absolute value:

Properties of Absolute Value

For all real numbers x ,

$$|x|\ge 0\,.$$

That is, absolute value is a nonnegative quantity.

For all real numbers x ,

$$|x|=|-x|\,.$$

That is, opposites have the same absolute value.

Is a minus sign inside or outside?

When working with absolute value expressions, you must pay attention to whether a minus sign is inside the absolute value or outside, because it makes a big difference!

A minus sign *outside* an absolute value denotes multiplication by -1. For example, $-|3| = (-1) \cdot |3| = (-1) \cdot 3 = -3$.

The position of the minus sign affects the order of operations:

- the expression '$|-3|$' denotes the sequence: start with the number 3, first take its opposite, and then find the resulting number's distance from zero.
- the expression '$-|3|$' denotes the sequence: start with the number 3, first find its distance from zero, and then multiply this result by -1.

Notice that 3, $|3|$, and $|-3|$ are all positive numbers.

However, -3, $-|3|$, and $-|-3|$ are all negative numbers.

EXERCISES

5. For these exercises, suppose that x is a nonzero number. That is, x might be positive and x might be negative, but x is not 0.

   Decide if each of the following is always positive (P), always negative (N), or sometimes positive/sometimes negative (SP/SN).

   If an expression is SP/SN, then give a value of x for which it is positive, and a value of x for which it is negative.

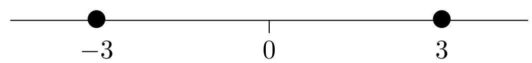
   (a) $|5|$
   (b) $|-5|$
   (c) $-|5|$
   (d) $-|-5|$
   (e) x
   (f) $|x|$
   (g) $-x$
   (h) $|-x|$
   (i) $2x$
   (j) $|2x|$
   (k) $-2x$
   (l) $|-2x|$
   (m) $-|2x|$
   (n) $-|-2x|$

Next, the mathematical word 'or' is reviewed, and then some new notation is introduced. This is preparation for solving sentences involving absolute value, which is the subject of the next section.

review: the mathematical word 'or'

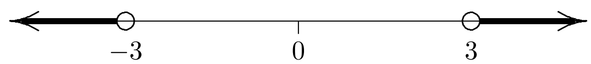
Recall that the 'or' sentence 'A or B' is true when at least one of the subsentences is true: when A is true, or B is true, or both A and B are true.

Thus, '$x = 3$ or $x = -3$' is true for the values of x shaded below:



You need to shade the value that makes the subsentence '$x = 3$' true, together with the value that makes the subsentence '$x = -3$' true.

As a second example, '$x > 3$ or $x < -3$' is true for the values of x shaded below:



You need to shade the values that makes the subsentence '$x > 3$' true, together with the values that makes the subsentence '$x < -3$' true.

EXERCISES

6. Shade the value(s) of x that make each sentence true:
   (a) $x = 2$
   (b) $x = -2$
   (c) $x = 2$ or $x = -2$
   (d) $x > 2$
   (e) $x < -2$
   (f) $x > 2$ or $x < -2$
   (g) $x = 2$ or $x \le -2$
   (h) $x = -2$ or $x \ge 2$
   (i) $x - 1 = 3$ or $x - 1 = -3$

the sentence '$x = \pm 2$' is shorthand for '$x = 2$ or $x = -2$'

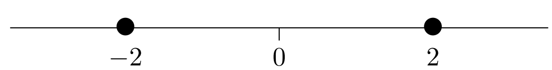
The sentence: '$x = \pm 2$'

is read as: 'x equals plus or minus 2'

and is a shorthand for: '$x = 2$ or $x = -2$'.

That is, the symbol '$\pm$' is short for 'plus or minus,' using the mathematical word 'or'.

Thus, the sentence '$x = \pm 2$' is true for the values of x shaded below:



EXERCISES

7. Shade the value(s) of x that make each sentence true:
   (a) $x = \pm 3$
   (b) $x = \pm 3$ or $x > 4$
   (c) $x = 0$ or $x = \pm 3$
   (d) $x = \pm 3$ or $x = \pm 4$

solving sentences like $2x - 1 = \pm 5$;

two possible approaches

When working with sentences involving '$\pm$', you have two choices: break into an 'or' sentence immediately, or wait until the last step to break into an 'or' sentence. Here's an example of each approach:

first approach: break into an 'or' sentence immediately

SOLVE: $2x - 1 = \pm 5$

FIRST APPROACH: Break into an 'or' sentence immediately:

$$\begin{aligned} 2x - 1 &= \pm 5 \\ 2x - 1 = 5 \quad &\text{or} \quad 2x - 1 = -5 \\ 2x = 6 \quad &\text{or} \quad 2x = -4 \\ x = 3 \quad &\text{or} \quad x = -2 \end{aligned}$$

second approach: wait until the last step to break into an 'or' sentence

SOLVE: $2x - 1 = \pm 5$

SECOND APPROACH: Wait until the last step to break into an 'or' sentence:

$$\begin{aligned} 2x - 1 &= \pm 5 \\ 2x &= \pm 5 + 1 \\ x &= \frac{\pm 5 + 1}{2} \\ x = \frac{5+1}{2} \quad &\text{or} \quad x = \frac{-5+1}{2} \\ x = 3 \quad &\text{or} \quad x = -2 \end{aligned}$$

you should understand both approaches; then, use whichever is most comfortable

You should understand both approaches. Then, use whichever approach is most comfortable for you. Notice that, in both approaches, you are writing a nice clean list of equivalent sentences. Also notice how you are using both the Addition and Multiplication properties of equality to replace an equation with an equivalent equation.

EXERCISES

8. Solve each sentence twice, using each of the two approaches just discussed. Which way is most comfortable for you?
   (a) $3x + 1 = \pm 2$
   (b) $2 - 5x = \pm 1$

next section: solving sentences involving absolute value

The next section presents tools for solving absolute value sentences such as these:

$$\begin{gathered} |x| = 5 \\ |x| < 5 \\ |x| > 5 \\ |x| \ge 5 \\ |2x - 1| \ge 5 \\ |3x - 5| < 4 \\ 5 - |1 + 3x| = 7 \end{gathered}$$

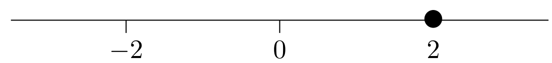
You'll see that the process is simple, providing you keep in mind the critical fact:

★★ **ABSOLUTE VALUE GIVES DISTANCE FROM** 0 ★★

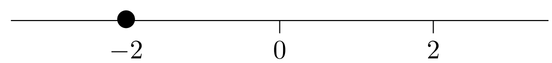
EXERCISES *web practice*	Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SOLUTIONS TO EXERCISES: INTRODUCTION TO ABSOLUTE VALUE

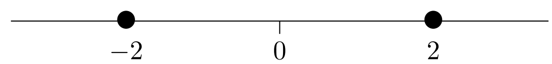
1. (a) all number(s) that are 2 units to the right of 0 :



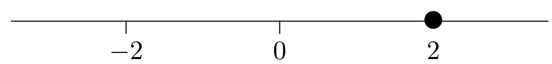
(b) all number(s) that are 2 units to the left of 0 :



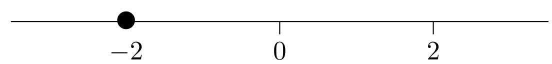
(c) all number(s) that are 2 units from 0 :



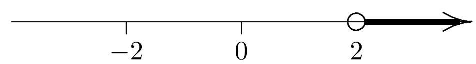
(d) all positive number(s) that are 2 units from 0 :



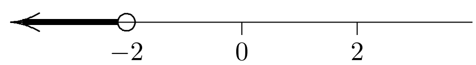
(e) all negative number(s) that are 2 units from 0 :



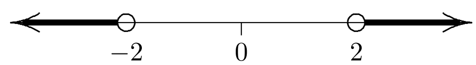
(f) all number(s) that are more than 2 units to the right of 0 :



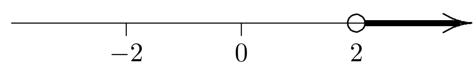
(g) all number(s) that are more than 2 units to the left of 0 :



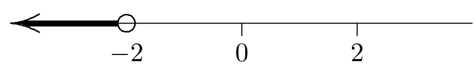
(h) all number(s) that are more than 2 units from 0 :



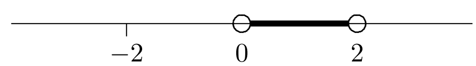
(i) all positive number(s) that are more than 2 units from 0 :



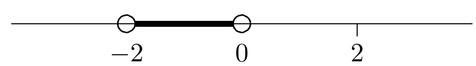
(j) all negative number(s) that are more than 2 units from 0 :



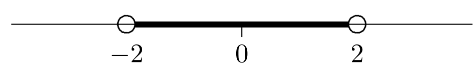
(k) all number(s) that are less than 2 units to the right of 0 :



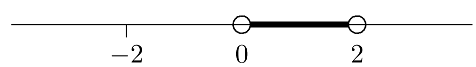
(l) all number(s) that are less than 2 units to the left of 0 :



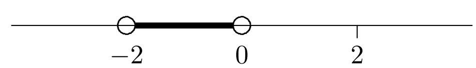
(m) all number(s) that are less than 2 units from 0 (include 0) :



(n) all positive number(s) that are less than 2 units from 0 :



(o) all negative number(s) that are less than 2 units from 0 :



2. (a) all numbers that are 3 units from 0

(b) all numbers that are more than 3 units from 0

(c) all numbers that are less than 3 units from 0

(d) all numbers that are k units from 0

(e) all numbers that are more than k units from 0

(f) all numbers that are less than k units from 0

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3. (a) The symbol $|4|$ represents the distance between 4 and 0. Thus, $|4| = 4$, since the number 4 is 4 units from zero.

(b) The symbol $|-4|$ represents the distance between -4 and 0. Thus, $|-4| = 4$, since the number -4 is 4 units from zero.

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4. (a) $|-7|$: the absolute value of 7; expression (number)

(b) $|1-3x|$: the absolute value of one minus three ex; expression (number)

(c) $|-3| > 2$: the absolute value of negative three is greater than two; sentence; T

(d) $|-3| < 2$: the absolute value of negative three is less than two; sentence; F

(e) $|-3| = 3$: the absolute value of three is equal to three; sentence; T

(f) $|3| = -3$: the absolute value of three is equal to negative three; sentence; F

(g) $|-3| = -3$: the absolute value of negative three is equal to negative three; sentence; F

(h) $|x| = 3$: the absolute value of ex is equal to three; sentence; ST/SF; true for $x = 3$ or $x = -3$, false otherwise

(i) $|x| = -3$: the absolute value of ex is equal to negative three; sentence; always false

---

5. (a) $|5|$ is positive ($|5| = 5$)

(b) $|-5|$ is positive ($|-5| = 5$)

(c) $-|5|$ is negative ($-|5| = -5$)

(d) $-|-5|$ is negative ($-|-5| = -5$)

(e) x is SP/SN; positive for $x = 3$, negative for $x = -3$

(f) $|x|$ is always positive

(g) $-x$ is SP/SN; positive for $x = -3$ ($-x = -(-3) = 3$) and negative for $x = 3$ ($-x = -3$)

(h) $|-x|$ is always positive

(i) $2x$ is SP/SN; positive for $x = 1$ and negative for $x = -1$

(j) $|2x|$ is always positive

(k) $-2x$ is SP/SN; positive for $x = -1$ ($-2x = -2(-1) = 2$) and negative for $x = 1$ ($-2x = -2(1) = -2$)

(l) $|-2x|$ is always positive

(m) $-|2x|$ is always negative

(n) $-|-2x|$ is always negative

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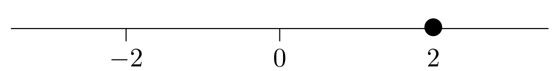
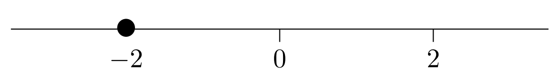
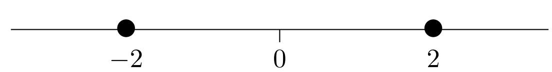
6. (a) $x = 2$



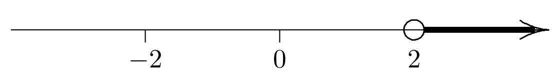
(b) $x = -2$



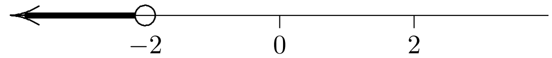
(c) $x = 2$ or $x = -2$



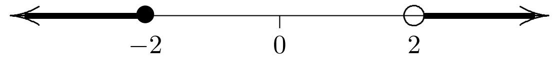
(d) $x > 2$



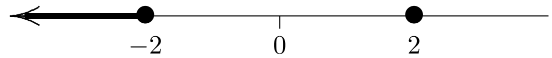
(e) $x < -2$



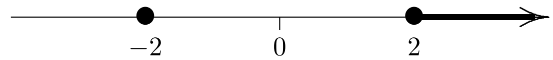
(f) $x > 2$ or $x \le -2$



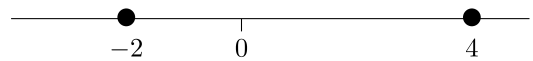
(g) $x = 2$ or $x \le -2$



(h) $x = -2$ or $x \ge 2$



(i) $x - 1 = 3$ or $x - 1 = -3$: rewrite as the equivalent sentence
$x = 4$ or $x = -2$:



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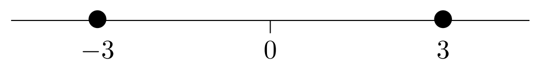
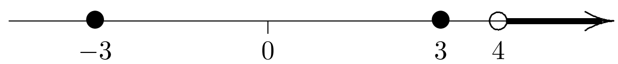
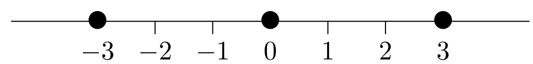
7. (a) $x = \pm 3$



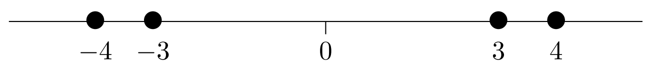
(b) $x = \pm 3$ or $x > 4$



(c) $x = 0$ or $x = \pm 3$



(d) $x = \pm 3$ or $x = \pm 4$



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8. (a)

$$\begin{gathered}
3x + 1 = \pm 2 \\
3x + 1 = 2 \quad \text{or} \quad 3x + 1 = -2 \\
3x = 1 \quad \text{or} \quad 3x = -3 \\
x = \frac{1}{3} \quad \text{or} \quad x = -1
\end{gathered}$$

$$\begin{gathered}
3x + 1 = \pm 2 \\
3x = \pm 2 - 1 \\
x = \frac{\pm 2 - 1}{3} \\
x = \frac{2 - 1}{3} \quad \text{or} \quad x = \frac{-2 - 1}{3} \\
x = \frac{1}{3} \quad \text{or} \quad x = -1
\end{gathered}$$

(b)

$$
\begin{gathered}
2-5x=\pm 1\\
-5x+2=\pm 1\\
-5x+2=1 \quad\text{or}\quad -5x+2=-1\\
-5x=-1 \quad\text{or}\quad -5x=-3\\
x=\frac{1}{5} \quad\text{or}\quad x=\frac{3}{5}
\end{gathered}
$$

$$
\begin{gathered}
2-5x=\pm 1\\
-5x+2=\pm 1\\
-5x=\pm 1-2\\
x=\frac{\pm 1-2}{-5}\\
x=\frac{1-2}{-5} \quad\text{or}\quad x=\frac{-1-2}{-5}\\
x=\frac{1}{5} \quad\text{or}\quad x=\frac{3}{5}
\end{gathered}
$$