31. TRANSFORMING TOOL #2 (the Multiplication Property of Equality)

a second transforming tool

In the previous section, we learned that adding/subtracting the same number to/from both sides of an equation makes the equation *look* different, but doesn't change its truth. This tool is used to 'transform' an equation into one that is easier to work with. A second transforming tool, the *multiplication property of equality*, is the subject of this section, and is stated below:

THEOREM
Multiplication Property of Equality

For all real numbers a and b, and for $c \neq 0$,

$$a = b \iff ac = bc\ .$$

EXERCISES

1. What is a 'theorem'?
2. What are the universal sets for a, b, and c in the previous theorem?
3. How would you read aloud the displayed sentence '$a = b \iff ac = bc$'? In particular, how do you read the symbol '$\iff$'?
4. Try to translate the theorem on your own! Think: what did you *do* to the equation '$a = b$' to get the equation '$ac = bc$'? What did you *do* to '$ac = bc$' to get the equation '$a = b$'?

two steps to take upon reading any theorem

Remember that when you first read *any* theorem, there are two questions that must be asked:

- What is the theorem telling you that you can *DO*?
  (That is, translate the theorem!)
- *WHY* is the theorem true? In particular, why isn't c allowed to equal 0?

These questions are addressed in the following paragraphs.

What is the theorem telling you that you can DO?

What is the theorem telling you that you can *DO*?

A first-level translation might go something like this:

> *No matter what real numbers are being held by a and b, and as long as c is not zero, then the compound sentence '$a = b \iff ac = bc$' will be true.*

Next, you must ask:

> *What does it mean for the compound sentence '$a = b \iff ac = bc$' to be true?*

It means that the subsentences '$a = b$' and '$ac = bc$' have precisely the same truth values. For *every* choice of real numbers a and b, and for *every* nonzero real number c:

If '$a = b$' is true, so is '$ac = bc$'.
If '$a = b$' is false, so is '$ac = bc$'.
If '$ac = bc$' is true, so is '$a = b$'.
If '$ac = bc$' is false, so is '$a = b$'.

OKAY: even though '$a = b$' and '$ac = bc$' may *look* different, they have the same truth. So what?

the critical part of the translation

This leads us to the critical part of the translation:

What did you *DO* to '$a = b$' to transform it into '$ac = bc$'?

Answer: You multiplied both sides by the nonzero number c.

$$\begin{tikzcd} a = b \arrow[rr, bend left, "\text{multiply by } c"] \arrow[rr, bend right, "\text{multiply by } c"'] & \iff & ac = bc \end{tikzcd}$$

Hence the first part of the translation:

> **You can multiply both sides of an equation by the same *nonzero* number, and this won't change the truth of the equation.**

Continuing the translation:

What did you *do* to '$ac = bc$' to transform it into '$a = b$'?

Answer: You divided both sides by the nonzero number c.

$$\begin{tikzcd} a = b & \iff & ac = bc \arrow[ll, bend right, "\text{divide by } c"'] \arrow[ll, bend left, "\text{divide by } c"] \end{tikzcd}$$

Hence the rest of the translation:

> **You can divide both sides of an equation by the same *nonzero* number, and this won't change the truth of the equation.**

the full translation of the Multiplication Property of Equality

Combining results, here's the way an instructor of mathematics might translate the Multiplication Property of Equality, to tell students what they can *do*:

> **You can multiply (or divide) both sides of an equation by the same *nonzero* number, and this won't change the truth of the equation.**

Since the *Multiplication* Property of *Equality* has to do with *multiplying* on both sides in a statement of *equality*, the name is appropriate.

★ *division isn't needed: everything can be done with multiplication*

Actually, division is superfluous: everything can be done with multiplication. That is, for all real numbers a, and for $b \neq 0$,

$$\frac{a}{b} = a \cdot \frac{1}{b} \,.$$

To divide by a number is the same as multiplying by its reciprocal. Thus, a translation of the previous theorem might simply be: 'You can multiply both sides of an equation by the same nonzero number, and this won't change the truth of the equation.' By stating that it works for multiplication, you're also stating that it works for division.

As with the Addition Property of Equality, there is a strong geometric reason as to why the Multiplication Property of Equality is true. Before discussing *why* it is true, however, let's understand why c can't equal 0, and use the theorem to solve some simple equations.

What goes wrong with multiplying or dividing by zero?

What goes wrong with multiplying or dividing by zero? That is, why isn't c allowed to equal zero in the Multiplication Property of Equality?

First of all, you will recall that division by zero is undefined; it's nonsensical; it's just not allowed. So zero certainly needs to be excluded when dividing.

But what about multiplying by zero? The problem is that multiplying by zero can *change* the truth of an equation: it can take a false equation to a true equation.

To see this, consider the false equation '$2 = 3$'. Multiplying both sides by zero results in the new equation '$2 \cdot 0 = 3 \cdot 0$' (that is, '$0 = 0$'), which is true.

$$\overbrace{2=3}^{\text{false}} \xrightarrow{\text{multiply by } 0} \overbrace{2\cdot 0 = 3 \cdot 0}^{\text{true}}$$

more precisely ...

More precisely, the compound sentence '$a = b \iff ac = bc$' might be false when c is zero. (Recall that a sentence of the form '$S1 \iff S2$' is false when the subsentences $S1$ and $S2$ have *different* truth values: when one is true, and the other is false.)

For example, choose $a = 2$, $b = 3$, and $c = 0$. For these choices, the compound sentence

$$a = b \iff ac = bc$$

becomes

$$\overbrace{\overbrace{2=3}^{\text{false}} \iff \overbrace{(2)(0) = (3)(0)}^{\text{true}}}^{\text{false}}\ .$$

The first subsentence is false; the second subsentence is true. Since the subsentences have different truth values, the compound sentence is false.

EXERCISES

5. It was shown above that multiplying by zero can take a false equation to a true equation. Can multiplying by zero take a true equation to a false equation? Comment.

6. Each compound sentence below corresponds to a particular choice for a, b, and c in the sentence '$a = b \iff ac = bc$'. In each case, identify the values that have been chosen for a, b, and c. Also, state whether the compound sentence is true or false. The first one is done for you.

(sample): $0 = 2 \iff 0 \cdot 5 = 2 \cdot 5$

Solution: $a = 0$, $b = 2$, $c = 5$.

The compound sentence is true:

$$\overbrace{\overbrace{0=2}^{\text{false}} \iff \overbrace{0 \cdot 5 = 2 \cdot 5}^{\text{false}}}^{\text{true}}$$

(a) $3 = 0 \iff 3 \cdot 5 = 0 \cdot 5$

(b) $0 = 0 \iff 0 \cdot 5 = 0 \cdot 5$

(c) $0 = 0 \iff 0 \cdot 0 = 0 \cdot 0$

(d) $0 = 1 \iff 0 \cdot 0 = 1 \cdot 0$

EXERCISE

7. Study your solutions to exercise (6), and answer the following questions:
   (a) If the compound sentence '$a = b \iff ac = bc$' is false, can you reach any conclusion about the value of c?
   (b) If the compound sentence '$a = b \iff ac = bc$' is true, can you reach any conclusion about the value of c?

★

extraneous solutions may arise when multiplying by zero

In the context of solving equations, 'multiplying by zero' can *add* a solution (a so-called **extraneous solution**). 'Adding a solution' means that a transformation was applied that took a false (or undefined) equation to a true equation. Students learn that whenever they have the *potential* of taking a false (or undefined) equation to a true equation, then they must check for extraneous solutions.

For example, a student using the familiar technique of 'cross-multiplying' might write down the following list of equations, and inadvertently conclude that the original equation is true when x is 0:

$$\begin{aligned} \frac{1}{x} &= \frac{2}{x} && \text{cross-multiply} \\ x &= 2x && \text{subtract } x \\ 0 &= x \\ x &= 0 \end{aligned}$$

In going from '$\frac{1}{x} = \frac{2}{x}$' to '$x = 2x$', the student multiplied both sides by x^2. (Note that $\frac{1}{x} \cdot x^2 = x$ and $\frac{2}{x} \cdot x^2 = 2x$.) This is fine as long as x is not zero. However, when x is 0, this operation took the undefined equation '$\frac{1}{0} = \frac{2}{0}$' to the true equation '$0 = 2 \cdot 0$'. Here, 0 is an extraneous solution.

extraneous solutions may arise when raising both sides of an equation to an even power

Another transformation that has the *potential* of taking a false (or undefined) equation to a true equation is raising both sides of an equation to an even power. For example, '$-1 = 1$' is false, but '$(-1)^2 = 1^2$' is true. Consequently, students learn that they must check for extraneous solutions when solving radical equations involving even roots.

Next, let's solve a simple equation using the Multiplication Property of Equality.

EXAMPLE
solving a simple equation

SOLVE: $\frac{x}{2} = 7$

Most people can solve this equation by inspection, because it's so simple. You need only think: 'What number, when divided by 2, gives 7?' The answer is of course 14.

However, let's solve it by using the Multiplication Property of Equality. We'll transform the original equation into one that's even *easier* to work with. The lines are numbered so that they can be easily referred to in the ensuing discussion. Recall that 'LHS' refers to the **L**eft-**H**and **S**ide of the equation; 'RHS' refers to the **R**ight-**H**and **S**ide of the equation.

line 1: $\frac{x}{2} = 7$ (Start with the original equation.)

line 2: $\overbrace{\frac{x}{2}}^{\text{previous LHS}} \cdot 2 = \overbrace{7}^{\text{previous RHS}} \cdot 2$ (Multiply both sides by 2.)

line 3: $x = 14$ (Simplify each side.)

The equations in lines 1, 2, and 3 all look different. That is, '$\frac{x}{2} = 7$' looks different from '$\frac{x}{2} \cdot 2 = 7 \cdot 2$' which looks different from '$x = 14$'. However, no matter what number is chosen for x, they all have the same truth values. The equation in line 3 is simplest. The only time that '$x = 14$' is true is when x is 14. Consequently, the only time that '$\frac{x}{2} = 7$' is true is when x is 14.

key ideas used in the previous example:

There are several key ideas (discussed below) used in transforming '$\frac{x}{2} = 7$' to '$x = 14$'. These key ideas are also used in a wide variety of similar transformations.

transform the equation to the form '$x = (\textit{some number})$'

- *TRANSFORM THE EQUATION TO THE FORM*

  $$`x = (\textit{some number})\text{'} .$$

  As discussed in the previous section, you want to end up with the variable all by itself on one side, and a specific number on the other side.

undo division with multiplication

- *UNDO DIVISION WITH MULTIPLICATION*. For example, to undo the operation 'divide by 2', you would apply the transformation 'multiply by 2'. Here's how this idea was used in the previous example:
  - The left-hand side started as '$\frac{x}{2}$', where x is being divided by 2.
  - We wanted x all by itself.
  - To undo 'divide by 2', the transformation 'multiply by 2' was applied.

undo multiplication with division

- *UNDO MULTIPLICATION WITH DIVISION*. For example, to undo the operation 'multiply by 5', you would apply the transformation 'divide by 5'.

The next exercise provides practice with these 'undoing' transformations.

EXERCISE

8. What would you *do* to each equation, to transform it to the form

$$`(\textit{some variable}) \;=\; (\textit{some number})\text{' ?}$$

The first one is done for you.

(sample) $2x = 5$

Answer: Divide both sides by 2. (To undo 'multiply by 2', apply the transformation 'divide by 2'.)

(a) $\frac{x}{4} = 10$
(b) $7x = 4$
(c) $11 = \frac{x}{7}$
(d) $6 = 7x$
(e) $3t = 5$
(f) $5 = 4t$

solving a more complicated equation

The real power comes when both the Addition and Multiplication Properties of Equality are used together. Study the next example, to prepare you for exercise 9. Observe that the solution process results in a list of equivalent equations, ending with one so simple that it can be solved by inspection.

SOLVE: $7x - 2 = 3x + 1$

SOLUTION: With the equation in this form, it is difficult to determine when it is true. You must think: 'What number, times 7, minus 2, is the same as the original number, times 3, plus 1?

So, we'll transform it into an equation that's easier to work with:

$$7x - 2 = 3x + 1$$

Start with the original equation.

$$\overbrace{7x-2}^{\text{previous LHS}} - 3x = \overbrace{3x+1}^{\text{previous RHS}} - 3x$$

First, you must get all the x's on the same side. (Here, the left-hand side was chosen.) To make the $3x$ 'disappear' from the right-hand side, undo 'add $3x$' with 'subtract $3x$'.

$$4x - 2 = 1$$

Simplify each side. Notice that seven ex, take away three ex, leaves four ex. Now, the variable x appears only on one side of the equation.

$$\overbrace{4x-2}^{\text{previous LHS}} + 2 = \overbrace{1}^{\text{previous RHS}} + 2$$

The expression '$4x-2$' on the left-hand side corresponds to the sequence of operations: 'take x, multiply by 4, then subtract 2.' The *last* operation is 'subtract 2'. Undo this *last* operation *first*, by adding 2 to both sides.

$$\begin{tikzcd} x \arrow[r, bend left, "\times 4"] & 4x \arrow[l, bend left, "\div 4"] \arrow[r, bend left, "-\ 2"] & 4x-2 \arrow[l, bend left, "+\ 2"] \end{tikzcd}$$

$$4x = 3$$

Simplify each side.

$$\frac{4x}{4} = \frac{3}{4}$$

We want only *one* ex ($1x$, which goes by the simpler name x) on the left-hand side. Undo 'multiply by 4' with 'divide by 4'.

$$x = \frac{3}{4}$$

Simplify the left-hand side.

the check

The final equation is true only when x is $\frac{3}{4}$ (three-fourths). Consequently, the original equation is true only when x is $\frac{3}{4}$. The following check requires arithmetic with fractions: use a calculator, if necessary.

$$7(\frac{3}{4}) - 2 \stackrel{?}{=} 3(\frac{3}{4}) + 1$$
$$\frac{21}{4} - 2 \stackrel{?}{=} \frac{9}{4} + 1$$
$$\frac{21}{4} - \frac{8}{4} \stackrel{?}{=} \frac{9}{4} + \frac{4}{4}$$
$$\frac{13}{4} = \frac{13}{4}$$

EXERCISE

9. Solve each equation. Use the Addition and Multiplication Properties of Equality, as needed. Check, using a calculator as needed. Use acceptable formats for both the solution and the check.

(a) $2x - 1 = 3 - 4x$

(b) $1 - x = 5x - 3$

Based on your answers to (a) and (b), are '$2x-1 = 3-4x$' and '$1-x = 5x-3$' equivalent equations? Justify your answer.

undoing the LAST operation FIRST

Suppose you're solving an equation (similar to the one in the previous example), and have gotten to the point where x appears on only one side. Now, you're trying to come up with transforming operations that will 'get x all by itself'. At this stage, it's important to think about what is being done to x, and then *undo the last operation first*.

For example, suppose you're solving the equation '$\frac{3x-5}{2} = 4$'. (Notice that x appears on only one side of the equation.) The expression $\frac{3x-5}{2}$ on the left-hand side corresponds to the following sequence of operations:

- start with x
- multiply by 3
- subtract 5
- divide by 2

$$x \xrightarrow{\times 3} 3x \xrightarrow{-\ 5} 3x-5 \xrightarrow{\div 2} \frac{3x-5}{2}$$

To 'undo' these operations and get back to x, each operation must be 'undone' in reverse order:

- multiply by 2
- add 5
- divide by 3

$$x \xleftarrow{\div 3} 3x \xleftarrow{+\ 5} 3x-5 \xleftarrow{\times 2} \frac{3x-5}{2}$$

Consequently, the solution process looks like this:

$$\begin{aligned} \frac{3x-5}{2} &= 4 & &\times 2 \\ 3x-5 &= 8 & &+ 5 \\ 3x &= 13 & &\div 3 \\ x &= \frac{13}{3} \end{aligned}$$

EXERCISE

10. First, list what is being done to x on the side of the equation where x appears. Then, solve each equation: 'undo' the operations in reverse order. Check your solutions. Use acceptable formats for both the solution and the check.

(a) $\frac{2x-3}{5} = 1$

(b) $2 = \frac{3x+1}{7}$

WHY is the Multiplication Property of Equality true?

Next, we'll discuss *why* the Multiplication Property of Equality is true. That is, *why* does multiplying or dividing both sides of an equation by the same nonzero number preserve the truth of the equation?

key ideas

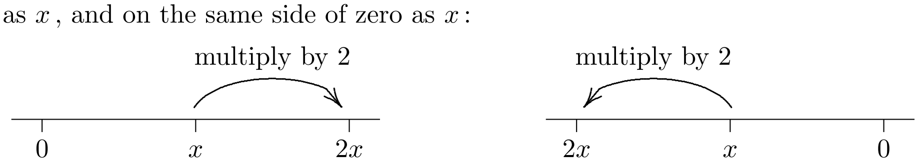
Several key ideas are needed:

- Recall the number-line interpretation of equality (and non-equality) of numbers: if '$a = b$' is true, then a and b live at the same place on a number line; if '$a = b$' is false, then a and b live at different places.
- Multiplying a real number x by another number causes a change of position on a number line: the particular way that x moves depends on what it's being multiplied by. Some sample cases are given next:

sample cases: movement resulting from multiplication by a a number

multiplying by 2

Multiplying x by 2 results in a new number, $2x$, which is twice as far from zero as x, and on the same side of zero as x:



multiplying by $\frac{1}{2}$

Multiplying x by $\frac{1}{2}$ results in a new number, $\frac{1}{2}x$, which is half as far from zero as x, and on the same side of zero as x:

multiplying by -2

Multiplying x by -2 results in a new number, $-2x$, which is twice as far from zero as x, but is on the opposite side of zero from x:

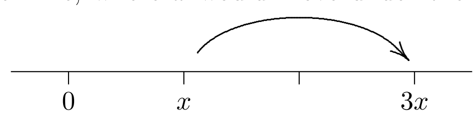
multiplying by $-\frac{1}{2}$

Multiplying x by $-\frac{1}{2}$ results in a new number, $-\frac{1}{2}x$, which is half as far from zero as x, but is on the opposite side of zero from x:

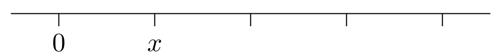
EXERCISE

11. Indicate, on each number line, where x would move under the specified operation.

Sample: Multiply x by 3

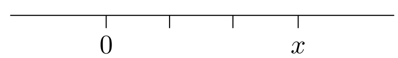


(a) Multiply x by 4

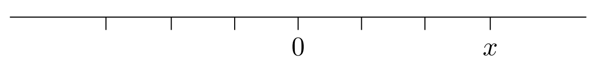


(b) Multiply x by -4

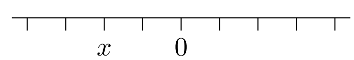
(c) Multiply x by $\frac{1}{3}$



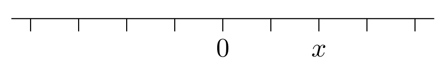
(d) Multiply x by $-\frac{1}{3}$



(e) Multiply x by 1.5 (one and one-half)



(f) Multiply x by -1.5



another key idea:

everything can be done with multiplication alone;
division isn't needed

Here is the final key idea.

- Understanding what happens when x is *multiplied* by another number is all that is needed: this covers everything that can happen with x is *divided* by another number.

Why is this? Remember that, for any nonzero real number x, the number $\frac{1}{x}$ is called the **reciprocal** of x. Also recall that *dividing by a number* is the same as *multiplying by the reciprocal of the number*:

$$x \overbrace{\div\ y}^{\text{dividing by } y} \quad \overbrace{=}^{\text{is the same as}} \quad x \overbrace{\cdot\ \frac{1}{y}}^{\text{multiplying by the reciprocal of } y}$$

More precisely (and using a horizontal fraction bar to denote division):

For all real numbers x, and for $y \neq 0$,

$$\frac{x}{y} = x \cdot \frac{1}{y} \quad .$$

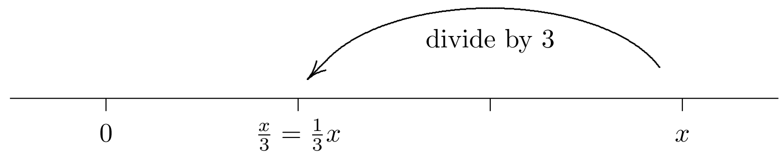
some examples:

dividing can be handled with an appropriate multiplication

Consider the following examples:

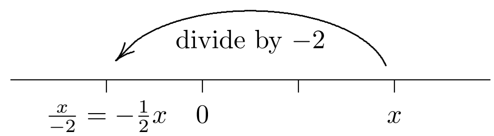
'Take x, and divide by 3' is the same as 'Take x, and multiply by $\frac{1}{3}$'.

The operation of dividing by 3 (or, equivalently, multiplying by $\frac{1}{3}$), results in a new number (called either $\frac{x}{3}$ or $\frac{1}{3}x$) which is one-third as far from zero as x, and on the same side of zero as x:



'Take x, and divide by -2' is the same as 'Take x, and multiply by $-\frac{1}{2}$'. (The reciprocal of -2 is $\frac{1}{-2}$, which goes by the simpler name $-\frac{1}{2}$.)

The operation of dividing by -2 (or, equivalently, multiplying by $-\frac{1}{2}$), results in a new number (called either $\frac{x}{-2}$ or $-\frac{1}{2}x$) which is half as far from zero as x, but on the opposite side of zero from x:



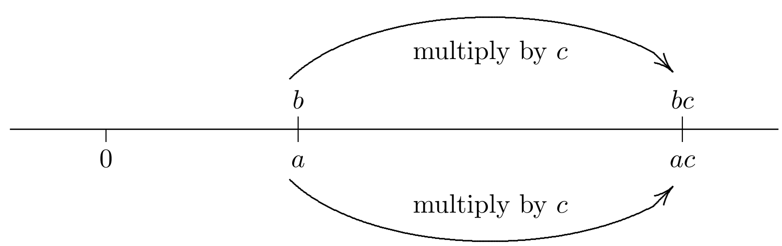
applying the key ideas:

if '$a = b$' is true, then '$ac = bc$' is true

Let's apply these key ideas to understanding the Multiplication Property of Equality. Remember: we're trying to show that the sentences '$a = b$' and '$ac = bc$' always have the same truth values (when $c \neq 0$).

- Suppose that '$a = b$' is true. Then, a and b live at the same place on a number line.
- Multiplying both numbers by c 'stretches' or 'shrinks' their original distance from zero (and causes a flip to the opposite side of zero, whenever c is negative).
- Since they started at the same place, they'll end up at the same place. Thus, '$ac = bc$' is also true.

The sketch below assumes that $c > 1$:

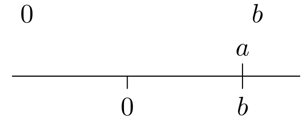


EXERCISE

12. Draw a picture, similar to the one above, showing what would happen for different values of c:

(a) c between 0 and 1 (like $c = \frac{1}{2}$)

(b) c between -1 and 0 (like $c = -\frac{1}{2}$)



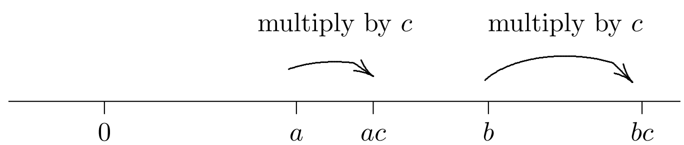
(c) c less than -1 (like $c = -2$)

if '$a = b$' is false, then '$ac = bc$' is false

One more time:

- Suppose that '$a = b$' is false. Then, a and b live at different places on a number line.
- Multiplying both numbers by c 'stretches' or 'shrinks' their original distance from zero (and causes a flip to the opposite side of zero, whenever c is negative).
- Since they started at different places, they'll end up at different places. Thus, '$ac = bc$' is also false.

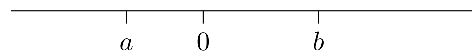
The sketch below assumes that $c > 1$:



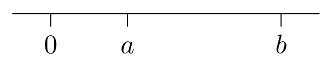
EXERCISE

13. Draw a picture, similar to the one above, showing what would happen to a and b when they are both multiplied by c:

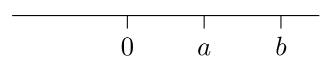
(a) $c > 1$ (like $c = 2$)



(b) c between 0 and 1 (like $c = \frac{1}{2}$)



(c) c between -1 and 0 (like $c = -\frac{1}{2}$)



(d) c less than -1 (like $c = -2$)

EXERCISE

14. Argue that if '$ac = bc$' is true, and c is not zero, then '$a = b$' is true.

15. Argue that if '$ac = bc$' is false, and c is not zero, then '$a = b$' is false.

You'll apply these transforming tools in the next section, and beyond!

In the next section, you'll gain extensive practice applying both of these new transforming tools to solve equations like these:

$2x - 5 = -3 + 7x\,,\qquad \frac{1}{2} - \frac{2}{3}x = \frac{3}{7}x + 5\,,\qquad 0.1 + 1.4x = 0.23x - 1$

Notice that there are only two types of terms—constant terms, and terms of the form kx. Such equations are called *linear equations in one variable*, and can be solved using only the addition and multiplication properties of equality. Onward!

END-OF-SECTION EXERCISES

For problems 16 and 17: SOLVE the equation. Use the Addition and Multiplication Properties of Equality, as needed. Check your solution. Use acceptable formats for both the solution and the check.

16. $t - 1 + 3t = 7 - 2t$

17. $\frac{5-2x}{7} = x$ (HINT: Clear fractions first, by multiplying both sides by 7.)

18. Show that the equation '$3x + 1 = 1 + 3x$' is equivalent to '$0 = 0$' (which is always true). What can you conclude about the equation '$3x + 1 = 1 + 3x$'?

19. Show that the equation '$3x + 1 = 3x + 2$' is equivalent to '$0 = 1$' (which is always false). What can you conclude about the equation '$3x + 1 = 3x + 2$'?

EXERCISES *web practice*	Go to my homepage **http://onemathematicalcat.org** and navigate to my Algebra I course, which has about 170 sequenced lessons. It can be used as a complete year-long high school course, or one semester in college. You're currently looking at the pdf version—you'll see that the HTML version has unlimited, randomly-generated, online and offline practice in every section. It's all totally free. Enjoy!

SECTION SUMMARY

TRANSFORMING TOOL #2

(the Multiplication Property of Equality)

NEW IN THIS SECTION	HOW TO READ	MEANING
the Multiplication Property of Equality		For all real numbers a and b, and for $c \neq 0$, $a = b \iff ac = bc$. Translation: multiplying or dividing both sides of an equation by the same *nonzero* number doesn't change the truth of the equation.
multiplying both sides of an equation by zero		Multiplying by zero can *change* the truth of an equation: it can take a false equation to a true equation. For example, '$2 = 3$' is false, but '$2 \cdot 0 = 3 \cdot 0$' (that is, '$0 = 0$') is true.
undoing the LAST operation FIRST		When trying to come up with transforming operations that will 'get x all by itself', it's important to think about what is being done to x, and then *undo the last operation first*. For example, suppose you're solving the equation '$\frac{3x-5}{2} = 4$'. The expression $\frac{3x-5}{2}$ on the left-hand side corresponds to the sequence of operations: take x, multiply by 3, subtract 5, divide by 2. To 'undo' these operations and get back to x, each operation must be 'undone' in reverse order: multiply by 2, add 5, divide by 3. $\begin{tikzcd} x \arrow[r, bend left, "\times 3"] & 3x \arrow[l, bend left, "\div 3"] \arrow[r, bend left, "-5"] & 3x-5 \arrow[l, bend left, "+5"] \arrow[r, bend left, "\div 2"] & \frac{3x-5}{2} \arrow[l, bend left, "\times 2"] \end{tikzcd}$

SOLUTIONS TO EXERCISES:
TRANSFORMING TOOL #2 (the Multiplication Property of Equality)

IN-SECTION EXERCISES:

1. A 'theorem' is a mathematical result that is both TRUE and IMPORTANT (★ that has been proved).

2. The universal set for both a and b is $\mathbb{R}$ (the set of real numbers). The universal set for c is all nonzero real numbers.

3. 'a equals b (*slight pause*) is equivalent to (*slight pause*) a times c equals b times c'

or

'a equals b if and only if a times c equals b times c'

4. You can multiply or divide both sides of an equation by any *nonzero* number, and this won't change the truth of the equation.

5. Suppose that '$a = b$' is true. Multiplying both sides by 0 gives the equation '$a \cdot 0 = b \cdot 0$', which is also true. Thus, multiplying by zero cannot take a true equation to a false equation.

6. (a) $a = 3\,,\quad b = 0\,,\quad c = 5$

The compound sentence is true:

$$\overbrace{\overbrace{3 = 0}^{\text{false}} \iff \overbrace{3 \cdot 5 = 0 \cdot 5}^{\text{false}}}^{\text{true}}$$

(b) $a = 0\,,\quad b = 0\,,\quad c = 5$

The compound sentence is true:

$$\overbrace{\overbrace{0 = 0}^{\text{true}} \iff \overbrace{0 \cdot 5 = 0 \cdot 5}^{\text{true}}}^{\text{true}}$$

(c) $a = 0\,,\quad b = 0\,,\quad c = 0$

The compound sentence is true:

$$\overbrace{\overbrace{0 = 0}^{\text{true}} \iff \overbrace{0 \cdot 0 = 0 \cdot 0}^{\text{true}}}^{\text{true}}$$

(d) $a = 0\,,\quad b = 1\,,\quad c = 0$

The compound sentence is false:

$$\overbrace{\overbrace{0 = 1}^{\text{false}} \iff \overbrace{0 \cdot 0 = 1 \cdot 0}^{\text{true}}}^{\text{false}}$$

7. (a) If the compound sentence is false, then c must equal 0. The *only* way that the sentences '$a = b$' and '$ac = bc$' can have different truth values, is when $c = 0$.

(b) If the compound sentence is true, then no conclusion can be reached about c. The number c might equal 0, or not.

8. (a) Multiply both sides by 4. (To undo 'divide by 4', apply the transformation 'multiply by 4'.)

(b) Divide both sides by 7. (To undo 'multiply by 7', apply the transformation 'divide by 7'.)

(c) Multiply both sides by 7. (To undo 'divide by 7', apply the transformation 'multiply by 7'.)

(d) Divide both sides by 7. (To undo 'multiply by 7', apply the transformation 'divide by 7'.)

(e) Divide both sides by 3. (To undo 'multiply by 3', apply the transformation 'divide by 3'.)

(f) Divide both sides by 4. (To undo 'multiply by 4', apply the transformation 'divide by 4'.)

9. Other orders of applying the transforming tools are possible. Only one format is illustrated.

(a)

$$\begin{aligned} 2x - 1 &= 3 - 4x \\ 2x &= 4 - 4x \\ 6x &= 4 \\ x &= \tfrac{4}{6} \\ x &= \tfrac{2}{3} \end{aligned}$$

CHECK:

$$\begin{aligned} 2(\tfrac{2}{3}) - 1 &\overset{?}{=} 3 - 4(\tfrac{2}{3}) \\ \tfrac{4}{3} - 1 &\overset{?}{=} 3 - \tfrac{8}{3} \\ \tfrac{4}{3} - \tfrac{3}{3} &\overset{?}{=} \tfrac{9}{3} - \tfrac{8}{3} \\ \tfrac{1}{3} &= \tfrac{1}{3} \end{aligned}$$

(b)

$$\begin{aligned} 1 - x &= 5x - 3 \\ 1 &= 6x - 3 \\ 4 &= 6x \\ \tfrac{4}{6} &= x \\ x &= \tfrac{2}{3} \end{aligned}$$

CHECK:

$$\begin{aligned} 1 - \tfrac{2}{3} &\overset{?}{=} 5(\tfrac{2}{3}) - 3 \\ \tfrac{3}{3} - \tfrac{2}{3} &\overset{?}{=} \tfrac{10}{3} - \tfrac{9}{3} \\ \tfrac{1}{3} &= \tfrac{1}{3} \end{aligned}$$

YES: For all real numbers x, $2x - 1 = 3 - 4x \iff 1 - x = 5x - 3$. Both equations are true only when $x = \frac{2}{3}$, and are false otherwise.

10. (a) Start with x, multiply by 2, subtract 3, divide by 5.

Undo with: multiply by 5, add 3, divide by 2.

$$\begin{tikzcd} x \arrow[r, bend left, "\times 2"] & 2x \arrow[l, bend left, "\div 2"] \arrow[r, bend left, "-3"] & 2x-3 \arrow[l, bend left, "+3"] \arrow[r, bend left, "\div 5"] & \dfrac{2x-3}{5} \arrow[l, bend left, "\times 5"] \end{tikzcd}$$

$$\begin{aligned} \tfrac{2x-3}{5} &= 1 \\ 2x - 3 &= 5 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

CHECK:

$$\begin{aligned} \tfrac{2(4)-3}{5} &\overset{?}{=} 1 \\ \tfrac{5}{5} &\overset{?}{=} 1 \\ 1 &= 1 \end{aligned}$$

(b) Start with x, multiply by 3, add 1, divide by 7.

Undo with: multiply by 7, subtract 1, divide by 3.

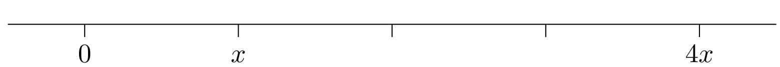
$$\begin{tikzcd} x \arrow[r, bend left, "\times 3"] & 3x \arrow[l, bend left, "\div 3"] \arrow[r, bend left, "+1"] & 3x+1 \arrow[l, bend left, "-1"] \arrow[r, bend left, "\div 7"] & \dfrac{3x+1}{7} \arrow[l, bend left, "\times 7"] \end{tikzcd}$$

$$\begin{aligned} 2 &= \tfrac{3x+1}{7} \\ 14 &= 3x + 1 \\ 13 &= 3x \\ \tfrac{13}{3} &= x \\ x &= \tfrac{13}{3} \end{aligned}$$

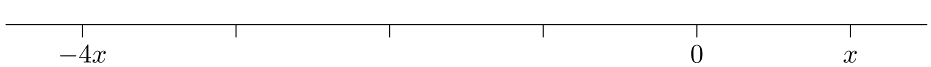
CHECK:

$$\begin{aligned} 2 &\overset{?}{=} \tfrac{3(\frac{13}{3})+1}{7} \\ 2 &\overset{?}{=} \tfrac{13+1}{7} \\ 2 &= 2 \end{aligned}$$

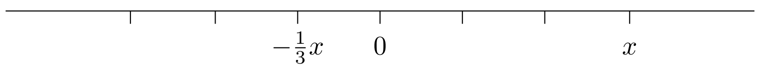
11. (a)

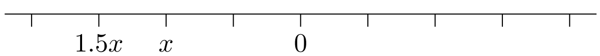


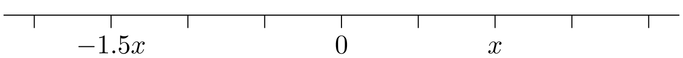
(b)

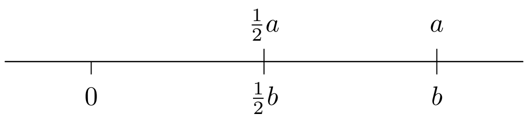


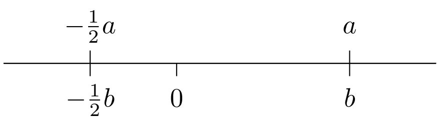
(c) 

(d) 

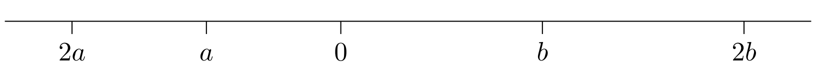
(e) 

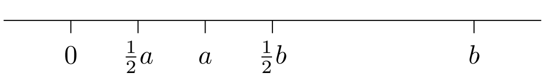
(f) 

12. (a) 

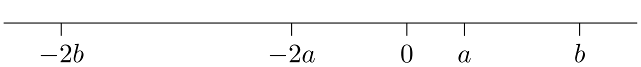
(b) 

(c) 

13. (a) 

(b) 

(c) 

(d) 

14. Suppose that '$ac = bc$' is true. Then, ac and bc live at the same place on a number line. Dividing by c (that is, multiplying by $\frac{1}{c}$), 'stretches' or 'shrinks' the original distance from zero (and causes a flip to the opposite side of zero, whenever c is negative). Since they started at the same place, they'll end up at the same place. Thus, '$a = b$' is also true.

15. Suppose that '$ac = bc$' is false. Then, ac and bc live at different places on a number line. Dividing by c (that is, multiplying by $\frac{1}{c}$), 'stretches' or 'shrinks' the original distance from zero (and causes a flip to the opposite side of zero, whenever c is negative). Since they started at different places, they'll end up at different places. Thus, '$a = b$' is also false.

END-OF-SECTION EXERCISES:

16.

$$\begin{aligned} t - 1 + 3t &= 7 - 2t \\ 4t - 1 &= 7 - 2t \\ 6t - 1 &= 7 \\ 6t &= 8 \\ t &= \tfrac{8}{6} \\ t &= \tfrac{4}{3} \end{aligned}$$

CHECK:

$$\begin{gathered} \tfrac{4}{3} - 1 + 3(\tfrac{4}{3}) \stackrel{?}{=} 7 - 2(\tfrac{4}{3}) \\ \tfrac{4}{3} - \tfrac{3}{3} + \tfrac{12}{3} \stackrel{?}{=} \tfrac{21}{3} - \tfrac{8}{3} \\ \tfrac{13}{3} = \tfrac{21}{3} \end{gathered}$$

17.

$$\begin{aligned} \tfrac{5-2x}{7} &= x \\ 5 - 2x &= 7x \\ 5 &= 9x \\ \tfrac{5}{9} &= x \\ x &= \tfrac{5}{9} \end{aligned}$$

CHECK:

$$\begin{gathered} \frac{5-2(\frac{5}{9})}{7} \stackrel{?}{=} \frac{5}{9} \\ \frac{\frac{45}{9}-\frac{10}{9}}{7} \stackrel{?}{=} \frac{5}{9} \\ \frac{\frac{35}{9}}{7} \stackrel{?}{=} \frac{5}{9} \\ \frac{35}{9}\cdot\frac{1}{7} \stackrel{?}{=} \frac{5}{9} \\ \frac{5}{9} = \frac{5}{9} \end{gathered}$$

18.

$$\begin{aligned} 3x + 1 &= 1 + 3x \\ 3x &= 3x \\ 0 &= 0 \end{aligned}$$

Since the final equation is always true, the original equation is always true. That is: for all real numbers x, $3x + 1 = 1 + 3x$.

19.

$$\begin{aligned} 3x + 1 &= 3x + 2 \\ 1 &= 2 \\ 0 &= 1 \end{aligned}$$

Since the final equation is always false, the original equation is always false. That is, the equation '$3x + 1 = 3x + 2$' is false, for all real numbers x.