ALGEBRA II OBJECTIVE: GR4

vertical translations (moving up and down): going from $y = f(x)$ to $y = f(x) \pm c$

horizontal translations (moving left and right): going from $y = f(x)$ to $y = f(x \pm c)$

DISCUSSION OF CONCEPT:

There are things that you can DO to an equation of the form $y = f(x)$ that will change the graph in a variety of ways. For example, you can move the graph up or down, left or right, reflect about the x or y axes, stretch or shrink vertically or horizontally.

An understanding of these transformations makes it easy to graph a wide variety of functions, by starting with a 'basic model' and then applying a sequence of transformations to change it to the desired function.

In this discussion, we will explore moving a graph up and down (vertical translations) and moving a graph left and right (horizontal translations).

When you finish studying this objective, you should be able to do a problem like this:

GRAPH $y = (x-3)^2 + 5$:

- Start with the graph of $y = x^2$. (This is the 'basic model'.)
- Add 5 to the previous y-values, giving the new equation $y = x^2 + 5$.
  This moves the graph UP 5 units.
- Replace every x by $x - 3$, giving the new equation $y = (x-3)^2 + 5$.
  This moves the graph to the RIGHT 3 units.

Here are ideas that are needed to understand graphical transformations.
First, some ideas regarding functions and the graph of a function:

- A function is a rule: it takes an input, and gives a unique output.
- If x is the input to a function f, then the unique output is called $f(x)$ (which is read as 'f of x').
- The *graph* of a function is a picture of *all* of its (input, output) pairs. We put the inputs along the horizontal axis (the x-axis), and the outputs along the vertical axis (the y-axis).
- Thus, the graph of a function f is a picture of all points of the form $(x, \overbrace{f(x)}^{y\text{-value}})$. Here, x is the input, and $f(x)$ is the corresponding output.
- The equation $y = f(x)$ is an equation in two variables, x and y. A solution is a choice for x and a choice for y that makes the equation true. Of course, in order for this equation to be true, y must equal $f(x)$.

  Thus, solutions to the equation $y = f(x)$ are points of the form $(x, \overbrace{f(x)}^{y\text{-value}})$.
- Compare the previous two ideas! You see that the requests 'graph the function f' and 'graph the equation $y = f(x)$' mean exactly the same thing.

  To "graph the function f" means to show all points of the form $(x, f(x))$.

  To "graph the equation $y = f(x)$" means to show all points of the form $(x, f(x))$.

Ideas regarding vertical translations (moving up/down):

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.
  Points on the graph of $y = f(x) + 3$ are of the form $(x, f(x) + 3)$.
  Thus, the graph of $y = f(x) + 3$ is the same as the graph of $y = f(x)$, shifted UP three units.
- **Transformations involving y work the way you would expect them to work—they are intuitive.**
- Here is the thought process you should use when you are given the graph of $y = f(x)$ and asked about the graph of $y = f(x) + 3$:

  original equation: $$y = f(x)$$

  new equation: $$\overbrace{y}^{\text{the new } y\text{-values}} \overbrace{=}^{\text{are}} \overbrace{f(x)}^{\text{the previous } y\text{-values}} + \overbrace{3}^{\text{with 3 added to them!}}$$

- Summary of vertical translations:
  Let p be a positive number.

  Start with the equation $y = f(x)$.
  Adding p to the previous y-values gives the new equation $y = f(x) + p$.
  This shifts the graph UP p units.
  A point (a, b) on the graph of $y = f(x)$ moves to a point $(a, b + p)$ on the graph of $y = f(x) + p$.

  Additionally:

  Start with the equation $y = f(x)$.
  Subtracting p from the previous y-values gives the new equation $y = f(x) - p$.
  This shifts the graph DOWN p units.
  A point (a, b) on the graph of $y = f(x)$ moves to a point $(a, b - p)$ on the graph of $y = f(x) - p$.

  This transformation type (shifting up and down) is formally called *vertical translation*.

Ideas regarding horizontal translations (moving left/right):

- Points on the graph of $y = f(x)$ are of the form $(x, f(x))$.

  Points on the graph of $y = f(x+3)$ are of the form $(x, f(x+3))$.

  How can we locate these desired points $(x, f(x+3))$?

  First, go to the point $(x+3, f(x+3))$ on the graph of $y = f(x)$.

  This point has the y-value that we want, but it has the wrong x-value.

  Move this point 3 units to the left. Thus, the y-value stays the same, but the x-value is decreased by 3. This gives the desired point $(x, f(x+3))$.

  Thus, the graph of $y = f(x+3)$ is the same as the graph of $y = f(x)$, shifted LEFT three units.

  Thus, replacing x by $x+3$ moved the graph LEFT (*not* right, as might have been expected!)

- **Transformations involving x do NOT work the way you would expect them to work—they are counter-intuitive—they are against your intuition.**

- Here is the thought process you should use when you are given the graph of $y = f(x)$ and asked about the graph of $y = f(x+3)$:

  original equation: $y = f(x)$

  new equation: $y = f(\overbrace{x+3}^{\text{replace } x \text{ by } x+3})$

  Replacing every x by $x+3$ in an equation moves the graph 3 units TO THE LEFT.

- Summary of horizontal translations:

  Let p be a positive number.

  Start with the equation $y = f(x)$.
  Replace every x by $x+p$ to give the new equation $y = f(x+p)$.
  This shifts the graph LEFT p units.
  A point (a, b) on the graph of $y = f(x)$ moves to a point $(a-p, b)$ on the graph of $y = f(x+p)$.

  Additionally:

  Start with the equation $y = f(x)$.
  Replace every x by $x-p$ to give the new equation $y = f(x-p)$.
  This shifts the graph RIGHT p units.
  A point (a, b) on the graph of $y = f(x)$ moves to a point $(a+p, b)$ on the graph of $y = f(x-p)$.

  This transformation type (shifting left and right) is formally called *horizontal translation.*

Notice that **different words** are used when talking about transformations involving y, and transformations involving x.

For transformations involving y (that is, transformations that change the y-values of the points), we say:

DO THIS to the previous y-value.

For transformations involving x (that is, transformations that change the x-values of the points), we say:

REPLACE the previous x-values by ....

In the following examples, the transformations could be applied in different orders to achieve the same results.

EXAMPLE:

State the transformations that take the graph of $y = f(x)$ to the graph of $y = f(x-2)+3$.

Equation	Action	Graphical Result
$y = f(x)$	(starting place)	
$y = f(x)+3$	add 3 to the previous y-values	move UP 3
$y = f(x-2)+3$	replace every x by $x-2$	move RIGHT 2

EXAMPLE:

State the transformations that take the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{x+5}-3$.

Equation	Action	Graphical Result
$y = \sqrt{x}$	(starting place)	
$y = \sqrt{x+5}$	replace every x by $x+5$	move LEFT 5
$y = \sqrt{x+5}-3$	subtract 3 from the previous y-values	move DOWN 3

EXAMPLE:

In this final example, be sure to notice that *every* x is replaced by $x-1$!

Equation	Action	Graphical Result
$y = x^2+2x$	(starting place)	
$y = x^2+2x+5$	add 5 to the previous y-values	move UP 5
$y = (x-1)^2+2(x-1)+5$	replace every x by $x-1$	move RIGHT 1